Studies on the Design of Robust Controllers for Linear Multivariable Discrete-Time Systems with Uncertainties(Dissertation_全文)

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STUDIES ON THE DESIGN OF ROBUST CONTROLLERS FOR LINEAR MULTIVARIABLE DISCRETE-TIME SYSTEMS WITH UNCERTAINTIES

A DISSERTATION
PRESENTED TO
KYOTO UNIVERSITY

by
KATSUMI MORIWAKI

July 1995

Acknowledgments

The author would like to express his sincere gratitude to Professor Tohru Katayama for his supervision and continuous encouragement in the preparation of this dissertation.

The author would also like to acknowledge the continuing guidance and encouragement of Dr. Hajime Akashi, Professor Emeritus of Kyoto University.

The author is also very grateful to Professor Koh-ichi Inoue, Professor Tsuneo Yoshikawa, Professor Hiroyuki Imai and Dr. Masao Adachi for their helpful discussions and suggestions.

Thanks are due to the author's many colleagues who have contributed to the realization of this dissertation. Last but not least the author wishes to express his appreciation to his wife Naomi and to their daughter Yuri for their love and patience.

K. Moriwaki

Abstract

This dissertation studies some problems of controller design for discrete-time linear multivariable systems. The dissertation is divided into three parts.

In the first part, it is considered that the design problems of state dead-beat controllers and output dead-beat controllers with asymptotic disturbance rejection by state feedback, and by dynamic compensation. It is well known that state feedback can provide arbitrary relocation of eigenvalues of a closed-loop system provided that the corresponding open loop system is reachable (or attainable). While the overall speed of response of the closed-loop system is determined by its eigenvalues, the shape of the transient response depends to a large extent on the closed-loop eigenvectors. Unlike the single-input case, one specification of closed-loop eigenvalues for a multi-input system does not define a unique closed-loop system. The generalized eigenstructure (eigenvalues and generalized eigenvectors) of a closed-loop system plays a fundamental role in the first part.

The concept of robust control with respect to the external disturbances is adopted to solve the dead-beat control problems in which there exists disturbance input. For the system with complete observation, some conditions are derived for the existence of a robust state feedback controller such that a finite time output dead-beat regulation can be made in spite of the existence of disturbance input. In order to design a dead-beat controller for the system with incomplete state observation, a method using a dynamic compensator is considered. A characterization of all robust dynamic dead-beat controllers such that the finite time state/output dead-beat regulation with asymptotic disturbance rejection occurs for arbitrary external disturbances is given.

The second part is concerned with adaptive control problems for linear multivariable discrete-time systems with unknown input which is interpreted as the state variable of the unknown input generator system with unknown parameters. For the output regulation problem of the augmented system, which is composed of the system to be controlled and the unknown input generator, the dynamic controller is derived. Using a linear functional observer and an adaptive state observer, it is shown that there exists a dynamic controller which can reduce the undesirable effect of the unknown input and stabilize the output of the system.

The adaptive controller is a control algorithm which is capable of initially tuning itself and of retuning itself in the event that the process characteristics subsequently change. There are identification errors in the estimated parameters from the beginning to the end of parameter estimation process. The response of the system output may be disturbed due to such adaptation errors at least until the adaptive algorithm converges successfully. For these systems, the dual controller is derived which implements efficient control actions by using two kinds of controllers – the stochastic sub-optimal regulator and the deterministic adaptive regulator – and switching them effectively.

In the third part, it is considered the decoupling problem of optimal control input for the system with two control agents. As the dimension of a dynamic system to be controlled becomes large, it is often too costly (sometimes practically impossible) to have only one decision maker (or controller) in the system who possesses all available information on the system and makes all the decisions for the system, for example, the optimal control input. The structure of the optimal control system with multiple controllers is derived, which is the globally equivalent to the optimal control system with a single controller.

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Chapter 1

INTRODUCTION

1.1 Historical Review

For many years, linear systems have been studied from several different points of view, in physics, mathematics, engineering, and many other fields. But the subject is such a fundamental and deep one that there is no doubt that linear systems will continue to be an object of study for as long as one can foresee. In recent engineering studies, finite-dimensional linear systems have been extensively studied, especially since the 1930s, the frequency-domain techniques that were commonly used often did not specially exploit the underlying finite-dimensionality of the system involved. Moreover, almost all this work was for single-input, single-output (or scalar) systems and did not seem to extend satisfactorily to multi-input, multi-output (or multivariable) systems that became increasing important in aerospace, process control and econometric applications in the 1950s.

The state-space description of linear systems, by which time-domain characteristics in control problems had deeply been considered, came into use in order to examine the structure of finite-dimensional linear systems, or linear dynamical systems as they often called, in more detail. The concept of reachability and observability was introduced by Kalman [38], using the state-space description of linear systems. The state-space formulation led to some new paradigms for system design and feedback compensation - pole assignment controllers, quadratic regulator synthesis, state observers and estimators, noninteracting control, robust controllers, and so on.

The fact that one can use state feedback to assign the closed-loop system poles any desired self-conjugate set of eigenvalues, provided that the open loop system is reachable, is a well-known and commonly used result. For single-input systems, this result is simple to derive and has known for some time. Eigenvalue placement in multi-input systems was studied by Popov [75], Wonham [86], and Simon and Mitter [82].

While the overall speed of response of the closed-loop system is determined by its eigenvalues, the shape of the transient response depends to a large extent on the closed-loop eigenvectors. Unlike the single-input case, the specification of closed-loop eigenvalues for multivariable systems does not define a unique closed-loop system. Hence, using the remaining degrees of freedom, Moore [54], Klein and Moore [42], and Sinswat and Fallside [83] presented an efficient method for designing a feedback controller to assign some eigenvectors besides eigenvalues.

Around 1970, Wonham and Morse [88] and Morse and Wonham [72] developed a method, called *a geometric approach*, for the solution of the problem of disturbance localization and decoupling control. The fundamental concept of this approach is that of *(A,B)-invariant subspace* and *(A,B)-controllability subspace*. The concept of *(A,B)-invariant subspace* was also discovered by Basile and Marro [10] together with the concept of *conditioned invariant subspace*. These concepts were subsequently shown to be applicable to a wide class of problems with the aid of pole assignment technique. Bhattacharyya, Pearson and Wonham [16] treated the problem of zeroing the output.

The disturbance localization problem of designing a controller such that the disturbance does not affect the output was solved by Wonham and Morse [88] by a state feedback. Chang and Rhodes [20] improved this result so that arbitrary pole assignment is achieved. Hamano and Furuta [32] solved this problem by an output feedback. The combined problem of decoupling/pole assignment and disturbance localization was studied by Chang and Rhodes [20] and Fabian and Wonham [26].

The discrete-time system models are widely used because the digital computers can be used as controllers, or measurement of the output are mostly available at discrete-time instants. A remarkable feature of discrete-time systems is the *dead-beat* performance, which was discovered by Kalman [38]. The dead-beat control problem is to design a controller which drives the system state to zero in finite number of steps from any nonzero initial state. It is known that such a dead-beat controller can be designed with a state feedback if the system is reachable. The dead-beat control problem can be regarded as a special case of pole assignment problem, because the dead-beat performance can be accomplished by assigning all the closed-loop system poles to zero.

The minimum time dead-beat control by a state feedback makes the transition matrix of the closed-loop system nilpotent with the minimum index. Rosenbrock [80] and Wonham and Morse [89] showed that this index is equal to the maximal value of the Kronecker invariants and that it is invariant under not only linear coordinate transformations but also transformations of state feedback type if the system is controllable. The minimum time dead-beat control was successfully used to control the temperature profile of a diffusion process by Leden [50]. Leden [48] also established connection between dead-beat control strategies and optimal control policy for linear, time-invariant, discrete-time systems where the performance index of the system is quadratic and only the terminal state of the system is penalized.

In order to design a dead-beat controller for time-invariant linear multivariable discrete-time systems with incomplete state observation, there are several methods to be considered, in which an output feedback is used with a dynamic compensator, or with an observer. Seraji [81] considered the problem of obtaining a dead-beat response for a discrete-time systems via constant output feedback and established the condition for the existence of a dead-beat controller by using the result of pole assignment by output feedback. This has the strong advantage in that any dynamic elements are not required and the state is taken to the origin in n steps instead of $2n$. But this approach is very limited and the way using some dynamical elements (a dynamic compensator or an observer) is more applicable. For reachable and observable systems, Porter and Bradshaw [77, 78] designed a dead-beat controller using an observer.

The design problem of dead-beat controllers with asymptotic disturbance rejection property for discrete-time systems has not been investigated so far. The existence condition of such a dead-beat controller is different from the solution of the problem of disturbance decoupling with stability(Wonham [85]), and there exists a system not to be disturbance decoupled from the initial time and yet to be disturbance decoupled asymptotically.

A large number of systems are subject to external disturbances or system parameter variations, which cause troubles under operation. Adaptive control has been considered as an alternative to design high performance control systems for the above problem formulations.

In the early 1950s there was extensive research on adaptive control, in connection with the design of autopilots for high performance aircrafts. Such aircrafts operate over a wide range of speeds and altitudes. It was found that ordinary constant-gain, linear feedback control could work well in one operating condition, but that change in operating conditions led to difficulties. A more sophisticated regulator, which could work well over a wide range of operating conditions, was therefore needed.

In the 1960s many contributions to control theory were important for the development of adaptive control. State space and stability theory were introduced. There were also important results in stochastic control theory. Dynamic programming, introduced by Bellman, increased the understanding of adaptive processes [12]. There were also major developments in system identification and in parameter estimation.

There was a renaissance of adaptive control in the 1970s, when different estimation schemes were combined with various design methods. The progress of the adaptive control theory and availability of microcomputers led to a series of successful applications in the late 1970s and early 1980s, resulting in many adaptive control algorithms - model reference adaptive control, adaptive observation, self-tuning regulator and so on (Landau [47]). In the same period correct proofs for stability of adaptive systems appeared, although under very restrictive assumptions([29]). Investigation of the necessity of those assumptions has sparked new and interesting research into robustness of adaptive control, as well as into controllers that are universally stabilizing(Åström [1]).

As the dimension of a dynamic system becomes large, it is often too costly (sometimes impossible) to have only one decision maker in the system. Hence, some literature has been devoted to the systems with more than one control agent. For a linear discrete-time dynamic system with two control agents, Aoki and Li [8] extended the results of Basile and Marro [9] to the system with two control agents, and gave a characterization of the set of state vectors which is compatible with the information possessed by an individual agent.

1.2 Organization of the Dissertation

The subject of this thesis is the study of controller design for time-invariant linear multivariable discrete-time systems. The thesis is divided into three parts.

In the first part, which consists of the following two chapters, we consider the problems of output dead-beat control with asymptotic disturbance rejection by state feedback(Chapter 2), and output dead-beat control with asymptotic disturbance

rejection by dynamic compensation(Chapter 3). The generalized eigenstructure (eigenvalues and generalized eigenvectors) of the closed-loop systems play a fundamental role in the first part.

The second part is concerned with adaptive control problems for linear multivariable discrete-time systems with unknown input. We develop a computer integrated control method to force the output from any initial state to zero as smooth as possible, in which two kinds of controllers are automatically changed. This part consists of Chapter 4 and Chapter 5.

The last part is concerned with optimal control problems with multiple controllers(Chapter 6). It is considered that the decoupling problem of linear optimal regulators with multiple control agents.

We first summarize the mathematical preliminaries in Section 1.3 and we outline the fundamental results of the control systems theory, especially, system description, dead-beat control, optimal control with respect to the quadratic performance criterion and adaptive control in Section 1.4.

Chapter 2 considers the design problem of output dead-beat controllers with asymptotic disturbance rejection by state feedback. The concept of robust controller with respect to disturbance input is defined and some conditions are derived for the existence of a robust state feedback controller such that a finite time output dead-beat regulation can be made independent of disturbance input. A design algorithm is obtained for computing a robust state feedback gain matrix which gives the required properties. Chapter 2 is based on H. Akashi, M. Adachi and K. Moriwaki [3], K. Moriwaki and H. Akashi [62, 69, 70, 71] and H. Akashi, K. Moriwaki and M. Shiho [6].

In Chapter 3, the design problem of dynamic output dead-beat controllers with asymptotic disturbance rejection is considered. As is the case of the state feedback systems, the choice of parameters of dynamic compensators, by which the extended state dead-beat performance is achieved, is, in general, not uniquely determined. There exists some freedom beyond specification of the closed-loop eigenvalues. Some conditions are derived for the existence of a robust dynamic output dead-beat controller for time-invariant linear multivariable discrete-time systems such that finite time output dead-beat regulation with asymptotic disturbance rejection occurs for arbitrary disturbance input. It is shown that the conditions obtained are different from the problem of disturbance decoupling with stability. Therefore, it is also shown that there exists a system not to be disturbance decoupled from the initial time yet to be disturbance decoupled asymptotically. Chapter 3 is based on K.

Moriwaki and H. Akashi [66] and H. Akashi, K. Moriwaki and M. Shiho [5].

In Chapter 4, the dynamic regulation problem for a linear multivariable discrete-time system with unknown input is considered. The unknown input is assumed to be generated from the unknown input generator system whose parameters are not known except its system's order. Using the unbiased linear functional observer and the stable adaptive state observer, the existence condition of the dynamic controller is examined which can reduce the undesirable effect of unknown input and stabilize the state of the system to be controlled. Chapter 4 is based on K. Moriwaki [57, 58, 59, 60], and K. Moriwaki and H. Akashi [61, 64, 67, 68].

Chapter 5 deals with a unified control problem for a linear multivariable discrete-time system with unknown input, where a computer integrated controller is implemented. In the control action, two kinds of controllers are effectively changed – (1)the time-varing stochastic sub-optimal regulator and (2)the time-invariant feedback controller with adaptive observer. Chapter 5 is based on K. Moriwaki [55, 56] and K. Moriwaki and H. Akashi [65].

In Chapter 6, the problem of control input decoupling in quadratic optimization is considered. For the quadratic optimization problem with two control agents, we consider how the optimal control input, which is given as the solution of optimal control problem with a single control agent, can be synthesized by two noninteractive optimal control agents, each of which satisfies the optimization problem of the corresponding subsystem. Chapter 6 is based on H. Akashi, M. Adachi and K. Moriwaki [2] and K. Moriwaki and H. Akashi [63].

Chapter 7 is a concluding chapter. We summarize the main results obtained in this dissertation, and then state several topics for further research.

1.3 Mathematical Preliminaries

Notation

If k is a positive integer, $\underline{k}$ denotes the set of integers $\{1, 2, 3, \ldots, k\}$. $\mathbf{R}^n$ denotes an n-dimensional Euclidean space. We consider only spaces over the field of real numbers or complex numbers. Linear spaces are denoted by script capitals, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, etc. Roman capitals stands both for maps (i.e., linear transformations) and their matrix representations. Unless otherwise stated, all spaces and subspaces

which appear explicitly are nonzero. All identity maps are represented by I. The dimension of $\mathcal{X}$ is written by $dim(\mathcal{X})$.

Let $C : \mathcal{X} \to \mathcal{Y}$ be a map. $\mathcal{X}$ is the *domain* of C and $\mathcal{Y}$ is the *codomain*; thus the size of matrix C is $dim(\mathcal{Y}) \times dim(\mathcal{X})$. The *kernel* (or *null space*) of C is the subspace

$$\operatorname{Ker} C := \{x : x \in \mathcal{X} \text{ and } Cx = 0\} \ (\subset \mathcal{X}), \tag{1.1}$$

while the *image* (or *range*) of C is the subspace

$$\begin{aligned} \operatorname{Im} C &:= \{y : y \in \mathcal{Y} \text{ and } {}^{\exists}x \in \mathcal{X},\, y = Cx\} \\ &= \{Cx : x \in \mathcal{X}\} \subset \mathcal{Y}. \end{aligned} \tag{1.2}$$

If $\mathcal{R} \subset \mathcal{X}$, we write

$$\begin{aligned} C\mathcal{R} &:= \{y : y \in \mathcal{Y} \text{ and } {}^{\exists}x \in \mathcal{R},\, y = Cx\} \\ &= \{Cx : x \in \mathcal{R}\}, \end{aligned} \tag{1.3}$$

and if $\mathcal{S} \subset \mathcal{Y}$,

$$C^{-1}\mathcal{S} := \{x : x \in \mathcal{X} \text{ and } Cx \in \mathcal{S}\}. \tag{1.4}$$

Both $C\mathcal{R} \subset \mathcal{Y}$ and $C^{-1}\mathcal{S} \subset \mathcal{X}$ are subspaces. Observe that C^{-1} is the *inverse image function* of the map C, and as such it will be regarded as a function from the set of all subspaces of $\mathcal{Y}$ to those of $\mathcal{X}$. In this usage C^{-1} does not denote a linear map from $\mathcal{Y}$ to $\mathcal{X}$. However, in the special case where $dim(\mathcal{X}) = dim(\mathcal{Y})$ and the ordinary inverse of C as a map $\mathcal{Y} \to \mathcal{X}$ happens to exist, this map will also be written, as usual, C^{-1}, since the two usages are then consistent and no confusion can arise.

As easy consequences of the definitions, we have

$$dim(C\mathcal{R}) = dim(\mathcal{R}) - dim(\mathcal{R} \cap \operatorname{Ker} C), \tag{1.5}$$

$$dim(C^{-1}\mathcal{S}) = dim(\operatorname{Ker} C) + dim(\mathcal{S} \cap \operatorname{Im} C), \tag{1.6}$$

and in particular, as $\operatorname{Im} C = C\mathcal{X}$,

$$dim(\mathcal{X}) = dim(\operatorname{Ker} C) + dim(\operatorname{Im} C). \tag{1.7}$$

Symbols are defined as follows,

A^T	:	the transpose of A,
$\mathbf{R}^{m\times n}$	:	the set of maps $\mathbf{R}^n \to \mathbf{R}^m$,
m.p. of A	:	minimal polynomial of A,
ch.p. of A	:	characteristic polynomial of A,
$\sigma(A)$	:	spectrum of A,
$\parallel A \parallel$	:	Euclidean norm of A,
$P_{\mathcal{O}}$	:	orthogonal projector on $\mathcal{O}$,
$\mathcal{O}^{\perp}$	:	annihilator of $\mathcal{O}$,
$\oplus$	:	direct sum,
$\emptyset$	:	empty set.

Rational Structure

Let $\mathcal{X}$ be a linear vector space with $dim(\mathcal{X}) = n$, and let $A : \mathcal{X} \to \mathcal{X}$ be a linear map. Write $\pi(\lambda)$ for the ch.p. of A. The *Hamilton - Cayley Theorem* states that $\pi(A) = 0$ ([30]). A polynomial is *monic* if the coefficient of its highest power of λ is 1. The m.p. of A is the monic polynomial $\alpha(\lambda)$ of least degree such that $\alpha(A) = 0$. The m.p. of A is unique and divides every nonzero polynomial $\beta(\lambda)$ such that $\beta(A) = 0$; in particular $\alpha(\lambda)$ divides $\pi(\lambda)$, so that $deg\alpha \leq n$, where $deg\alpha$ is the degree of $\alpha(\lambda)$.

If $\alpha(\lambda) = \pi(\lambda)$ *i.e.*, $deg\alpha = n$, A is said to be *cyclic*, and there exists $g \in \mathcal{X}$ such that the vectors g, Ag, ..., $A^{n-1}g$ form a basis for $\mathcal{X}$ (Wonham [85]).

Moore-Penrose Pseudoinverse

Let $\mathcal{X}$ and $\mathcal{Y}$ be linear vector spaces and let $A : \mathcal{X} \to \mathcal{Y}$ be a linear map. The Moore-Penrose pseudoinverse of A, written A^+, is the linear map $A^+ : \mathcal{Y} \to \mathcal{X}$ uniquely defined by the following four axioms(Boullion and Odell [17], Rao and Mitra [79]).

$$\begin{array}{ll} AA^+A = A, & A^+AA^+ = A^+, \\ (AA^+)^T = AA^+, & (A^+A)^T = A^+A. \end{array} \tag{1.8}$$

Using the Moore-Penrose pseudoinverse A^+, the following relations are frequently utilized in the present thesis(Ben-Israel and Greville [13]),

$$\begin{array}{rcll} (A^T)^+ & = & (A^+)^T, & \\ A^+ & = & (A^TA)^+A^T = A^T(AA^T)^+, & \\ A^+ & = & (A^TA)^{-1}A^T, & \text{for } A : \text{monic}, \\ A^+ & = & C^T(CC^T)^{-1}(B^TB)^{-1}B^T, & \text{for } A = BC \text{ and } B, C^T : \text{monic}, \end{array} \tag{1.9}$$

$$\begin{array}{ll} P_{\mathrm{Im}A} = AA^+, & P_{\mathrm{Ker}A^T} = P_{(\mathrm{Im}A)^\perp} = I - AA^+, \\ P_{\mathrm{Im}A^T} = A^+A, & P_{\mathrm{Ker}A} = P_{(\mathrm{Im}A^T)^\perp} = I - A^+A, \end{array} \tag{1.10}$$

$$P_{\mathcal{S}} = P_{\mathcal{V}} + P_{\mathcal{W}} \text{ for } \mathcal{V} \perp \mathcal{W} \text{ and } \mathcal{V} + \mathcal{W} = \mathcal{S}, \tag{1.11}$$

$$\begin{array}{rcccl} \mathrm{Ker}\, A^+ & = & (\mathrm{Im}\, A^T)^\perp & = & \mathrm{Ker}\, A, \\ \mathrm{Im}\, A^+ & = & (\mathrm{Ker}\, A)^\perp & = & \mathrm{Im}\, A^T. \end{array} \tag{1.12}$$

A necessary and sufficient condition for the matrix equation $AXB = C$ to have a solution is

$$AA^+CB^+B = C, \tag{1.13}$$

in which the general solution is

$$X = A^+CB^+ + Z - A^+AZBB^+, \tag{1.14}$$

where Z is an arbitrary matrix(Rao and Mitra [79]).

Algebra of Linear Space

Assume that $\mathcal{X}$ and $\mathcal{Y}$ are linear vector spaces and that $C : \mathcal{X} \to \mathcal{Y}$ is a linear map. For any linear vector spaces $\mathcal{R}$ ($\subset \mathcal{X}$) and $\mathcal{S}$ ($\subset \mathcal{Y}$), it can be shown that the following relations hold (Wonham [85]).

$$\begin{array}{rcl} C(C^{-1}\mathcal{S}) & = & \mathcal{S} \cap \mathrm{Im}\, C, \\ C^{-1}(C\mathcal{R}) & = & \mathcal{R} + \mathrm{Ker}\, C, \end{array} \tag{1.15}$$

$$\begin{array}{rcl} C\mathcal{R} \subset \mathcal{S} & \Leftrightarrow & \mathcal{R} \subset C^{-1}\mathcal{S} \\ \mathcal{S} = A^{-1}\mathcal{R} & \Leftrightarrow & \mathcal{S}^\perp = A^T\mathcal{R}^\perp \end{array} \tag{1.16}$$

Let $A : \mathcal{X} \to \mathcal{X}$ be a linear map. Let $\mathcal{V}$ be a subspace of $\mathcal{X}$ such that $A\mathcal{V} \subset \mathcal{V}$. Then, the subspace $\mathcal{V}$ is said to be an *A-invariant* subspace. The class of *A-invariant* subspaces of $\mathcal{X}$ is denoted by $\underline{\mathcal{I}}(A\,;\,\mathcal{X})$, that is,

$$\underline{\mathcal{I}}(A\,;\,\mathcal{X}) := \{\,\mathcal{V} \,:\, A\mathcal{V} \subset \mathcal{V}\,,\ \mathcal{V} \subset \mathcal{X}\,\}\,. \tag{1.17}$$

Furthermore, for an arbitrary subspace $\mathcal{G} \subset \mathcal{X}$, $\underline{\mathcal{I}}(A\,;\,\mathcal{G})$ denotes the subclass of *A-invariant* subspaces contained in $\mathcal{G}$:

$$\underline{\mathcal{I}}(A\,;\,\mathcal{G}) := \{\,\mathcal{V} \,:\, A\mathcal{V} \subset \mathcal{V}\,,\ \mathcal{V} \subset \mathcal{G}\,,\ \mathcal{G} \subset \mathcal{X}\,\}\,. \tag{1.18}$$

The following two propositions are known (Wonham [85]).

Proposition 1.1 *The class of subspaces $\underline{\mathcal{I}}(A\,;\,\mathcal{X})$ is closed under the operation of subspace addition.*

The subspace $\mathcal{V} \subset \mathcal{X}$ is said to be an *(A, B)-invariant* subspace if it is *A-invariant (mod B)*, that is, $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$ where $\mathcal{B} := \operatorname{Im} B$. The class of *(A, B)-invariant* subspaces of $\mathcal{X}$ is denoted by $\underline{\mathcal{I}}(A\,,\,B\,;\,\mathcal{X})$. For an arbitrary subspace $\mathcal{K} \subset \mathcal{X}, \underline{\mathcal{I}}(A\,,\,B\,;\,\mathcal{K})$ denotes the subclass of *(A, B)-invariant* subspaces contained in $\mathcal{K}$:

$$\underline{\mathcal{I}}(A\,,\,B\,;\,\mathcal{K}) := \{\,\mathcal{V} \,:\, A\mathcal{V} \subset \mathcal{V} + \mathcal{B}\,,\ \mathcal{V} \subset \mathcal{K}\,,\ \mathcal{K} \subset \mathcal{X}\,\}\,. \tag{1.19}$$

Remark. It is not true in general that the property of *A-invariance* is preserved by subspace intersection.

If $\underline{\mathcal{V}}$ is a family of subspaces of $\mathcal{X}$, we can define the *largest* or *supremal* element $\mathcal{V}^*$ as an element of $\underline{\mathcal{V}}$, when it exists. Thus, $\mathcal{V}^* \in \underline{\mathcal{V}}$, and if $\mathcal{V} \in \underline{\mathcal{V}}$ then $\mathcal{V} \subset \mathcal{V}^*$. It is clear that $\mathcal{V}^*$ is unique.

Proposition 1.2 *Let $\underline{\mathcal{V}}$ be a nonempty class of subspaces of $\mathcal{X}$, closed under addition. Then, $\underline{\mathcal{V}}$ contains a supremal element $\mathcal{V}^*$.*

1.4 Control Systems

Linear Discrete-Time Systems

Consider a finite dimensional linear time-invariant multivariable discrete-time system (or a plant):

$$x(k+1) = Ax(k) + Bu(k)\,, \tag{1.20}$$

$$y(k) = Cx(k)\,, \tag{1.21}$$

$$z(k) = Dx(k)\,,\ k = 0, 1, 2, \ldots, \tag{1.22}$$

where $x \in \mathbf{R}^n$ is the state of the system to be controlled, $u \in \mathbf{R}^r$ the control input, $y \in \mathbf{R}^{m_y}$ the measurement output, $z \in \mathbf{R}^{m_z}$ the regulated output, and $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times r}, C \in \mathbf{R}^{m_y\times n}, D \in \mathbf{R}^{m_z\times n}$ are the constant matrices, respectively. The linear space of the state vectors x is described by $\mathcal{X}$. In the same way, $\mathcal{U}$ and $\mathcal{Y}$ denote linear spaces of the control input vectors u and the measurement output vectors y, respectively. The discrete-time system (1.20)–(1.22) is illustrated in Figure 1.1, where the delay operator q^{-1} is used to obtain a more compact expression for the difference equations describing discrete-time models in the time domain. In the study of designing control systems or control systems' synthesis, one of the main objectives is to determine the *Control Mechanism*, in which the data of the measurement output are processed and the control input is tuned so that the prescribed specification are satisfied. For the delay operator, the following relations hold.

$$q^{-1}x(k) = x(k-1)\,,\ q^{-d}x(k) = x(k-d)\,.$$

Using the initial state $x_0 := x(0)$ and the control input sequence

$$\{\,u(0),\ u(1),\ u(2),\ \ldots,\ u(k-1)\,\},\quad \text{for } 0 < k < \infty\,,$$

the system state $x(k)$ is given by

$$x(k) = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} Bu(i)\,. \tag{1.23}$$

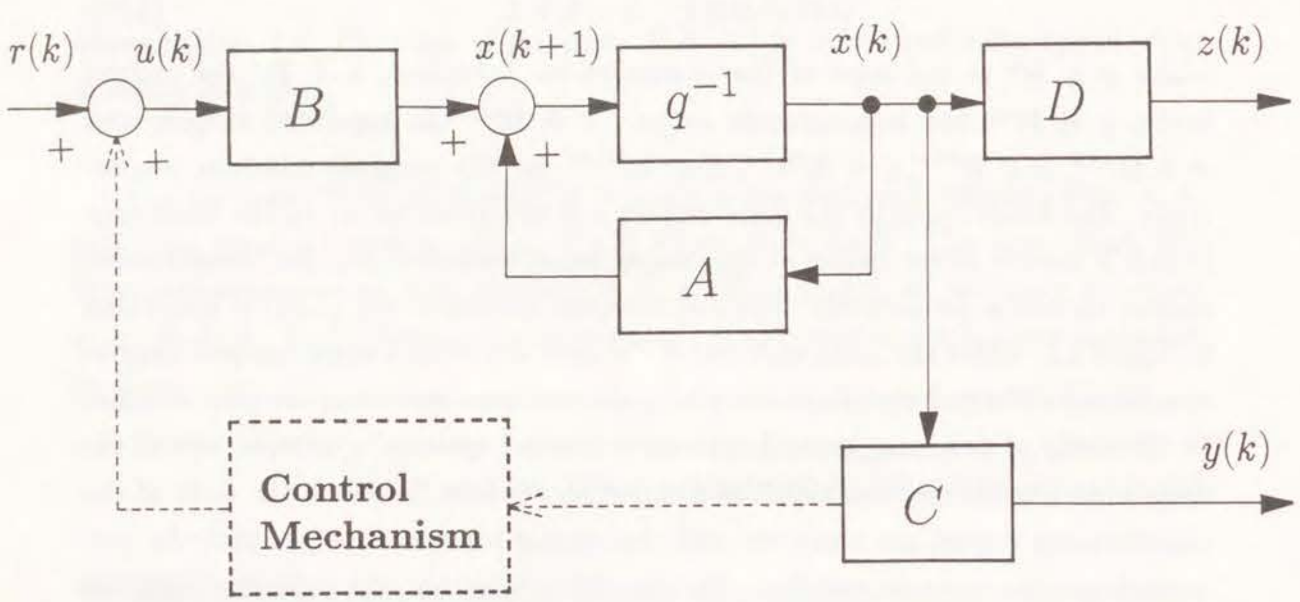

Figure 1.1: Block diagram of a linear Discrete-Time System.

A state $x(k_r)$ is *reachable from* x_0 if there exist k_r $(0 < k_r < \infty)$, and $u(i)$ $(i \in \underline{k_r - 1})$, such that $x(k_r)$ is described by (1.23). Let $\mathcal{R}_0$ be the set of states reachable from $x_0 = 0$. Then, it is readily checked $\mathcal{R}_0$ is a linear space of $\mathcal{X}$. For the system (1.20), the following subspaces are defined :

$$\langle A|\mathcal{B}\rangle := \mathcal{B} + A\mathcal{B} + A^2\mathcal{B} + \cdots + A^{n-1}\mathcal{B} . \tag{1.24}$$

where $\mathcal{B} := \operatorname{Im} B$.

Proposition 1.3 *(Wonham [85])*

$$\mathcal{R}_0 = \langle A|\mathcal{B}\rangle . \tag{1.25}$$

The subspace $\mathcal{R}_0 = \langle A|\mathcal{B}\rangle \subset \mathcal{X}$ is the *controllable subspace* of the pair $(A\,,\,B)$. From (1.24) (and the Hamilton-Cayley theorem [30]), it is clear that $A\mathcal{R}_0 \subset \mathcal{R}_0$, that is, $\mathcal{R}_0$ is A-invariant. It is easy to see that $\mathcal{R}_0$ is the smallest A-invariant subspace containing $\mathcal{B}$.

Definition 1.1 *The pair $(A\,,\,B)$ is called <u>reachable</u> if its controllable subspace is the whole space, that is,*

$$\langle A|\mathcal{B}\rangle = \mathcal{X}. \tag{1.26}$$

The system (1.20),(1.21) is called *observable* if there exists an integer $p > 0$ such that given

$$\{\, u(0),\, u(1),\, u(2),\, \ldots,\, u(p-1) \,\}$$

and

$$\{\, y(0),\, y(1),\, y(2),\, \ldots,\, y(p-1) \,\},$$

it is possible to deduce x_0.

Definition 1.2 *The pair $(C\,,\,A)$ is called <u>observable</u> if*

$$\bigcap_{i=1}^{n} Ker(CA^{i-1}) = 0, \tag{1.27}$$

where $n := dim(\mathcal{X})$.

Definition 1.3 *The system (1.20),(1.21) or the triplet (C, A, B) is called complete or (minimal) if the pair (C, A) is observable and the pair (A, B) is reachable.*

Dead-Beat Control

The dead-beat controllers drive the state or the output of a discrete-time system to zero in a finite number of steps. Early contributions to the problem of dead-beat control were given by Kalman et al. [38, 39]. Kalman solves the problem of transferring the state of a single-input sampled-data system from any initial state to equilibrium with zero error in a minimum number of time steps. The solution was given as a linear state feedback. In solving this problem Kalman introduces the concepts of reachability and observability. The possibility designing a sampled-data regulator in such a way that the error in the response to a step input is identically zero in the sampling instants after a certain number of time steps was first pointed out by Bergen and Ragazzini [14].

Bertram and Sarachik [15] relaxed the minimality condition, which is introduced by Kalman, and demanded that the output of the system from any initial state should be transferred to the origin, possibly with a small error, in a given a number of time steps. Instead the control strategies were derived so as to satisfy other desiderata, for instance, minimizing the energy consumption required for the transfer. The strategies were control programs applicable to multivariable systems. Dead-beat regulators that force the state of a multivariable system to zero were first derived by Farison et al. [28] and Kučera [45]. The problem of constructing a linear state feedback which forces the output of a single-input, single-output system from its initial state to zero in a minimum number of time steps was considered by Kučera in [44].

The dead-beat regulators are intimately connected to canonical structures of linear systems. In fact such regulators which force the state of a multivariable system to the origin can be obtained directly from a controllable canonical form of the system(Luenberger [52]). However, there exist no canonical structure which give the corresponding controllers for the output case. The computation of the canonical structures which yield the dead-beat regulators is numerically ill-conditioned for systems with a large number of state variables. The output dead-beat controllers may give an unstable closed-loop system, therefore, it is important to consider constrained output dead-beat controllers which always give a stable closed-loop system(Leden [49]).

Optimal Control(Anderson and Moore [7])

For the linear system (1.20) and (1.21), the optimal control problem is to choose the control input $u(\cdot)$ so as to minimize

$$J(u) = x^T(N)Q_N x(N) + \sum_{i=0}^{N-1} y^T(i)Qy(i) + \sum_{i=0}^{N-1} u^T(i)Ru(i) \,, \qquad (1.28)$$

where Q_N and Q are positive semi-definite symmetric matrices, and R is positive definite symmetric matrix. The optimum solution will be shown to be

$$u^{opt}(k) = L(k)x(k) \,, \qquad (1.29)$$

where

$$L(k) = -[R + B^T\Phi(k+1)B]^{-1}B^T\Phi(k+1)A \,, \qquad (1.30)$$

and $\Phi(\cdot)$ obeys a Riccati-type difference equation

$$\begin{aligned} \Phi(k) &= A^T\Phi(k+1)A + C^TQC \\ &\quad - A^T\Phi(k+1)B[R + B^T\Phi(k+1)B]^{-1}B^T\Phi(k+1)A \,, \\ &\quad (k = N, N-1, \ldots, 1, 0), \end{aligned} \qquad (1.31)$$

with terminal condition

$$\Phi(N) = Q_N \,. \qquad (1.32)$$

The optimal cost can be calculated as

$$J^{opt} = x_0^T\Phi(0)x_0 \,. \qquad (1.33)$$

Adaptive Control

There exist several techniques for designing adaptive controllers. We shall describe the main characteristics of the two widely recognized adaptive control families — *Direct* and *Indirect* adaptive schemes (Canudas de Wit [19]).

Adaptive control schemes based on *direct* (or *implicit*) approach have to find a control law, function of the measurable system states, seeking to minimize a predefined error which is a quantifier of the closed-loop system performance. The controller's parameters are then adjusted on-line until the tracking or model error is nullified(Figure 1.2).

In *indirect* or *explicit* adaptive control schemes, the plant is estimated explicitly by some on-line estimation procedures. The controller design is based on a model with unknown parameters. Subsequently, the controller's parameters are updated using the model provided by the estimation block as shown in Figure 1.3.

There exist important differences between the *direct* and the *indirect* control schemes. For instance, while for the former the parameter convergence is not necessary for achieving closed-loop stability, it is essential requirement for the latter. In other words, in *direct* adaptive schemes the stability analysis can be executed by using any of the standard nonlinear system analysis techniques such as Lyapunov functions, functional analysis. On the other hand, in *indirect* method in order to conclude closed-loop stability it is necessary to assume that the model parameters converge to the true plant parameters. A certain degree of excitation in the internal plant states is then necessary to achieve this last assumption. To some extent, the *direct* adaptive controllers generates their own source of excitation, whereas the *indirect* adaptive schemes requires extended excitation, such as noise, or reference input.

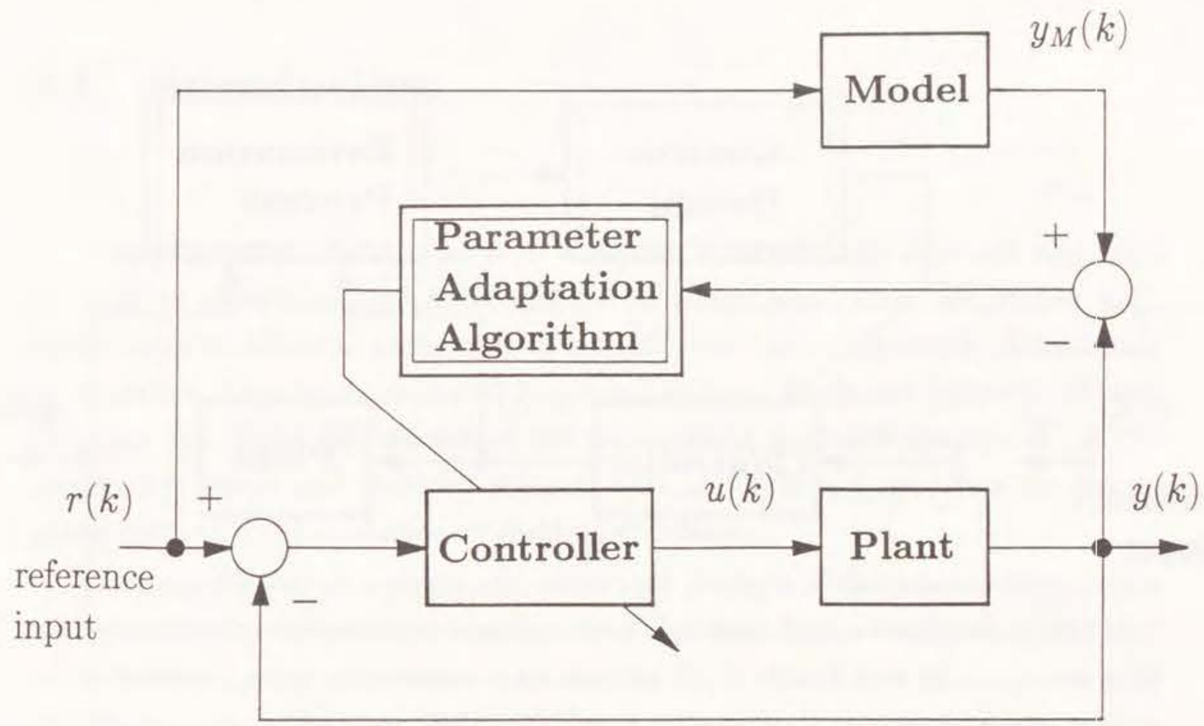

Figure 1.2: Direct adaptive control scheme.

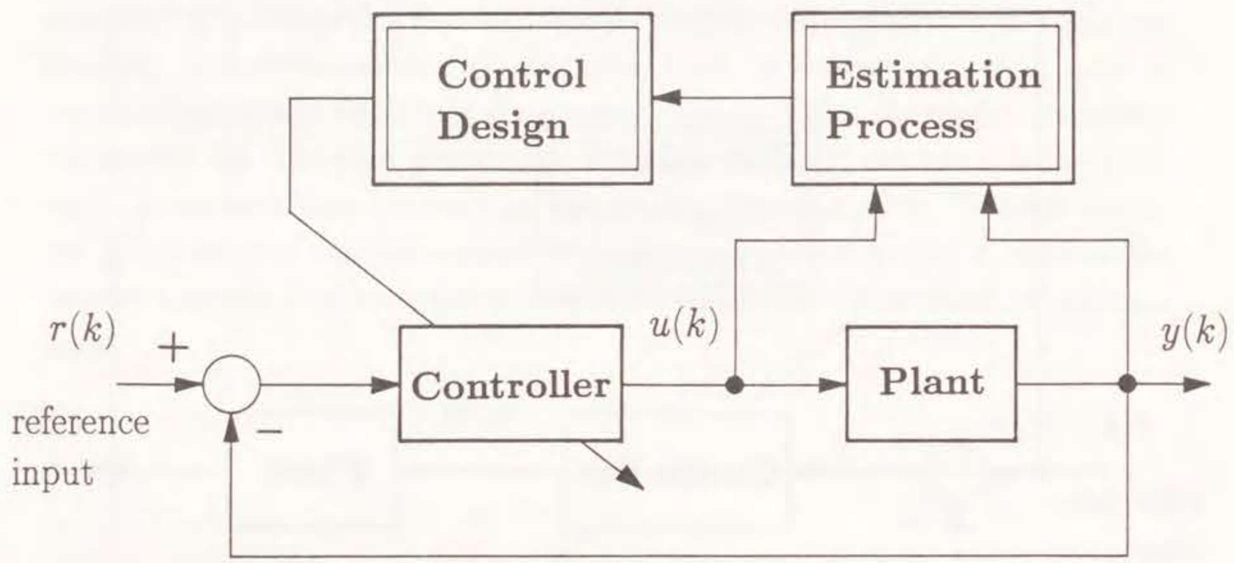

Figure 1.3: Indirect adaptive control scheme.

Chapter 2

DEAD-BEAT CONTROLLERS BY STATE FEEDBACK

2.1 Introduction

The purpose of this chapter is to construct a robust state feedback controller by using an eigenvalue-generalized eigenvector assignment (Klein and Moore [42]). Disturbance localization problem with simultaneous pole assignment, stabilization or regulation, have been considered by several authors. Basile and Marro [9, 11] and Wonham and Morse [88] discovered the fundamental geometric concept of (A, B)-invariant subspace and obtained the necessary and sufficient condition for disturbance decoupling (disturbance localization) problem.

Wonham [85] solved a practically important problem of disturbance localization with stability. In the problem formulation of the disturbance localization with stability, however, many requirements are met for the feedback gain as compared with the freedom offered by state feedback. Hence, it is difficult to find the solution even if the existence of the solution is ensured.

In order to avoid such a difficulty, the robust controller(Davison [23, 24], Davison and Wang [25]) with respect to disturbance input is adopted here and some conditions are derived for the existence of a robust state feedback controller for a linear time-invariant, multivariable discrete-time system such that a finite time output dead-beat regulation can be achieved in the presence of disturbance input.

Moore [54] has shown that for a distinct self-conjugate set of eigenvalues, the additional freedom offered by state feedback, beyond pole placement, is employed for the selection of a set of eigenvectors from an allowable class. The allowable class was characterized and an algorithm was given to compute a feedback matrix which gives the selected eigenvalue-eigenvector sets. Klein and Moore [42] extended these results in order to characterize the class of generalized eigenvector chains which can be obtained with a given set of nondistinct eigenvalues. For continuous-time system, Davison [23] considered the asymptotic decoupling problem with disturbance rejection, in which the conditions obtained were different from the decoupling problem(Falb and Wolovich [27]).

The rest of this chapter is organized as follows. In Section 2.2, some preliminary results and definitions are given. In Section 2.3, the concept of robust control for discrete-time systems is adopted in order to solve the output dead-beat problem in the presence of disturbance input and a characterization of all such robust controllers with respect to disturbance input is provided. A modified algorithm of Klein and Moore [42] is obtained for computing a robust state feedback matrix which gives the required properties in Section 2.4. We discuss the difference between the problem of dead-beat control with asymptotic disturbance rejection and the problem of simultaneous disturbance localization and pole-assignment in Section 2.5. Some numerical examples are given in Section 2.6.

2.2 Problem Statement and Preliminaries

Consider the time-invariant linear multivariable discrete-time system represented by

$$x(k+1) = Ax(k) + Bu(k) + Es(k)\ ,\ \ x(0) = x_0\ , \tag{2.1}$$

$$z(k) = Dx(k)\ ,\ \ k = 0,1,2,3,\ldots, \tag{2.2}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $z \in \mathbf{R}^{m_z}$ and $s \in \mathbf{R}^{n_s}$ denote the state, control input, regulated output and disturbance input, respectively. Figure 2.1 illustrates the open-loop control system (2.1)-(2.2). Employing the state feedback law of the form

$$u(k) = Fx(k)\ ,\ \ k = 0,1,2,\ldots, \tag{2.3}$$

the resulting closed-loop system of (2.1) (Figure 2.2) is

$$x(k+1) = (A+BF)x(k) + Es(k)\ . \tag{2.4}$$

It is assumed that the system pair (A, B) is reachable and that the matrix B has rank t $(\leq r)$, that is,

$$\mathrm{rank}\,[\,B\ AB\ A^2B\ \cdots\ A^{n-1}B\,] = n\ , \tag{2.5}$$

$$\mathrm{rank}\,B = t\ ,\ \ t \leq r\ . \tag{2.6}$$

It is also assumed that

$$\mathrm{Im}\,E\ \subset\ \mathrm{Ker}\,D\ . \tag{2.7}$$

Since the pair (A, B) is reachable, there exists a state feedback matrix F such that all closed-loop spectra of $A+BF$ can be assigned freely. The problem treated here is the following.

[**Robust Output Dead-Beat Control Problem by State Feedback**]

Given the system (2.1) and (2.2), find all constant feedback gain F such that

$$z(T_f + k) = 0\ ,\ \ k = 0,1,2,\ldots, \tag{2.8}$$

in which the number of steps T_f being minimum, in the presence of disturbance input $s(k)$ $(0 \leq k < \infty)$.

If such a constant feedback gain F exists, the controller is called a *robust output dead-beat controller with respect to input disturbances.* In Chen [21], it is shown that if the closed-loop system (2.4) has t blocks of order $d_1, d_2, \ldots, d_t$ in its Jordan canonical form (with no loss of generality, we can assume $d_1 \geq d_2 \geq \ldots \geq d_t \geq 1$ and $\sum_{i=1}^{t} d_i = n$), then there are t corresponding generalized eigenvector chains $\{v_{ij} | j \in \underline{d_i}, i \in \underline{t}\}$ defined by

$$\begin{aligned} (A+BF)v_{i1} &= 0\ , \\ (A+BF)v_{ij} &= v_{ij-1},\ j = 2,3,\ldots,d_i. \end{aligned} \tag{2.9}$$

From (2.5) and index order assumption, we have

$$(A+BF)^{d_1} = 0\ . \tag{2.10}$$

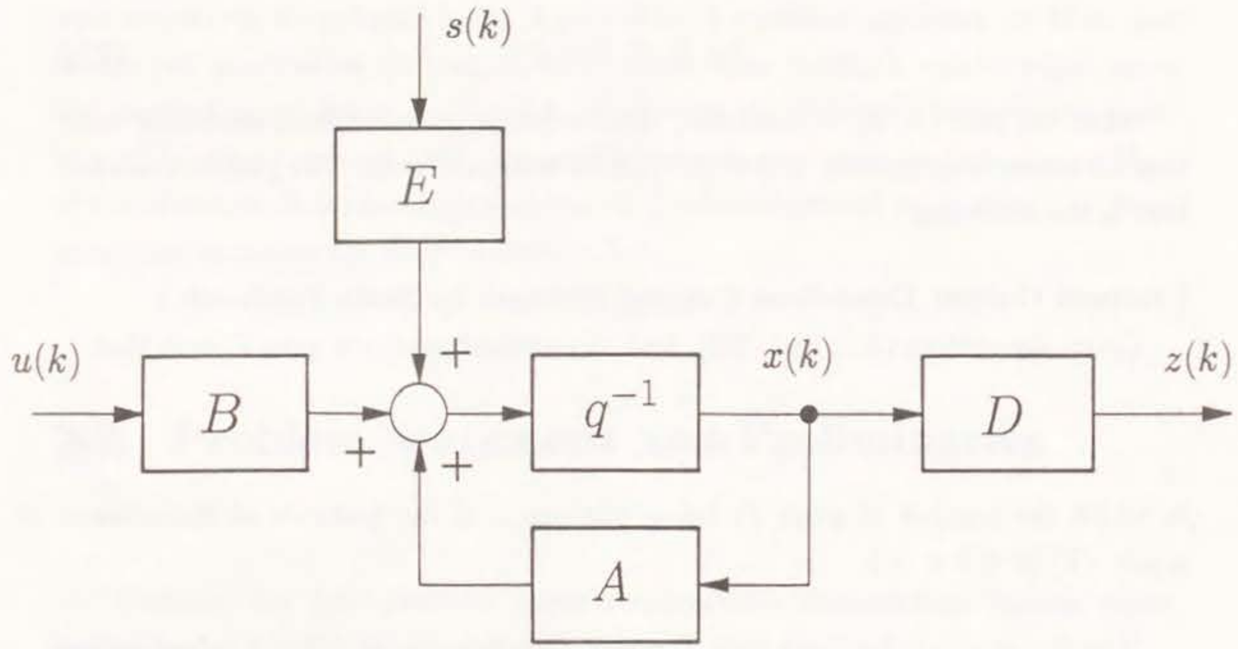

Figure 2.1: Block diagram of an open-loop control system.

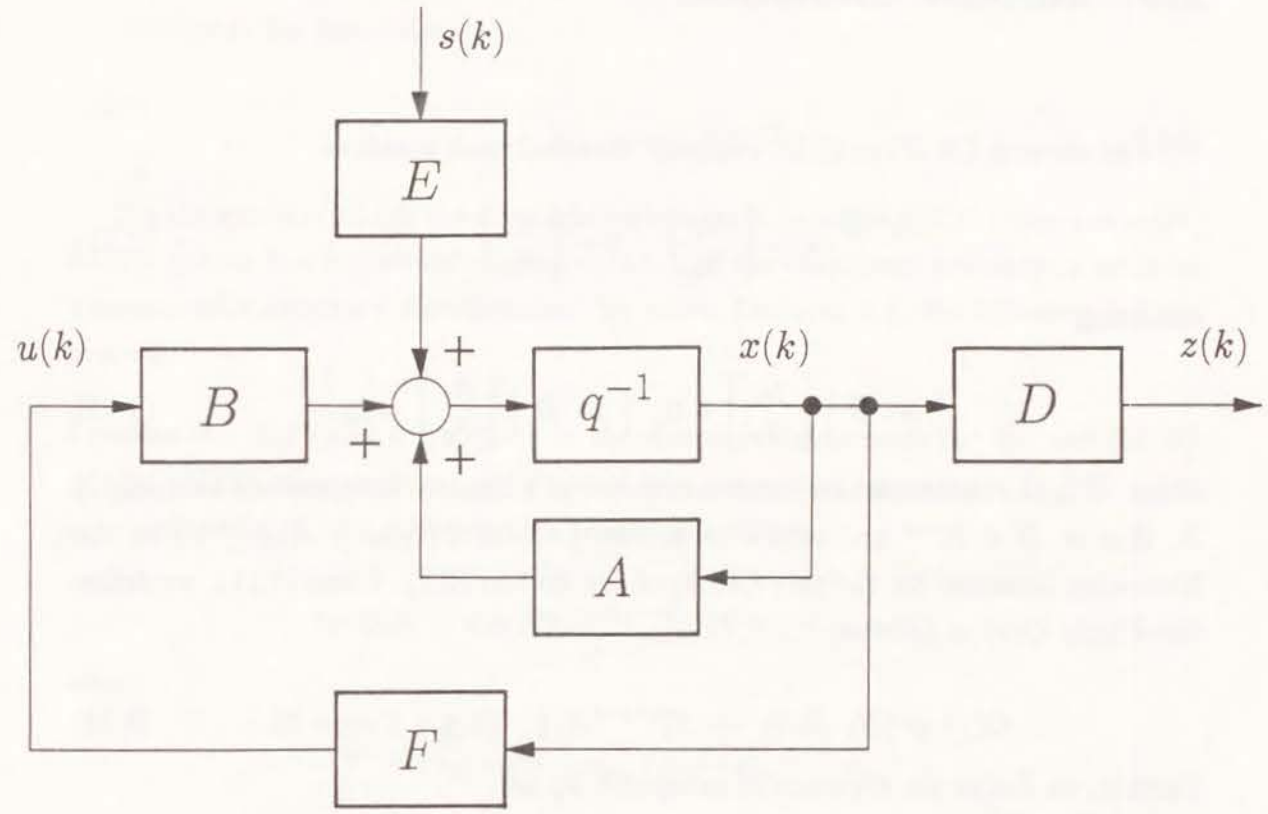

Figure 2.2: Block diagram of a closed-loop control system.

Therefore, for the given index set $\{d_i | i \in \underline{t}\}$, we can construct a constant feedback control gain F with the number of steps d_1 as the minimum state settling steps. Recalling the definition and properties of the *Kronecker invariants*(Kalman [37], Popov [76]), if the Kronecker invariants of the system pair (A, B) of (2.1) are $\kappa_i (i \in \underline{t})$ with $\kappa_i \geq \kappa_{i+1} \geq 1$, one concludes that the choice of κ_1 as T_f gives the minimum possible index of nilpotency.

2.3 Robust Controller

For the pair (A, B) in (2.1), compute maximal rank matrices

$$N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \tag{2.11}$$

satisfying

$$\begin{bmatrix} A & \overline{B} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = 0, \quad \begin{bmatrix} A & \overline{B} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = I_n. \tag{2.12}$$

where $\overline{B}$ is the matrix whose columns consist of t linearly independent columns in B, that is, $\overline{B} \in \mathbf{R}^{n \times t}$ and $\operatorname{rank} \overline{B} = t$. Let $\{\kappa_i, i \in \underline{t}\}$ $(\kappa_i \geq \kappa_{i+1} \geq 1)$ be the Kronecker invariant for the pair (A, B) of the system (2.1). Using (2.11), we define the matrix $Q(j)$ as follows :

$$Q(j) := [\, N_1 \;\; S_1 N_1 \;\; \cdots \;\; S_1^{\kappa_1 - j - 2} N_1 \,], \quad (0 \leq j \leq \kappa_1 - 2). \tag{2.13}$$

Further, we define the sequence of subspaces $\mathcal{Y}_\mu$ as

$$\begin{aligned} \mathcal{Y}_0 &= \operatorname{Im} E, \\ \mathcal{Y}_1 &= A\mathcal{Y}_0 + \mathcal{B}, \\ \mathcal{Y}_i &= A\mathcal{Y}_{i-1} + \mathcal{Y}_{i-1}, \\ & \quad (i = 2, 3, \ldots, \kappa_1 - j, \; 0 \leq j \leq \kappa_1 - 2). \end{aligned} \tag{2.14}$$

By using (2.13) and (2.14), we obtain the following theorem for the existence of such a *robust output dead-beat controller* that (2.8) is achieved independent of input disturbances.

Theorem 2.1 *Let $\{\kappa_i | i \in \underline{t}, \; t \leq r\}$ be the Kronecker invariants for the pair (A, B) of (2.1). There exists a robust output dead-beat controller which drives the*

output $z(k)$ of (2.2) from any initial state to zero at most in κ_1-steps independent of input disturbances and thereafter keeps the output zero in the class of controllers that drive the output $z(k)$ from any initial state to zero in κ_1-steps, if the following two conditions are satisfied.

1. *There exists a nonnegative integer τ_0 satisfying the following condition :*

$$\tau_0 = \max\{\, j \,|\, \operatorname{Im} Q(j) \supset \operatorname{Im} E, \; 0 \leq j \leq \kappa_1 - 2 \,\}. \tag{2.15}$$

2. *For the integer τ_0 of (2.15), the subspace $\mathcal{Y}_{\kappa_1 - \tau_0 - 2}$, which is defined by (2.14), satisfies the following :*

$$\mathcal{Y}_{\kappa_1 - \tau_0 - 2} \subset \operatorname{Ker} D. \tag{2.16}$$

For the system (2.1), if the nonnegative integer τ_0 satisfying (2.15) does not exist, such a system has no robust controller that has the dead-beat property in spite of unmeasurable arbitrary disturbances. To prove Theorem 2.1, the following lemma is used.

Lemma 2.1 *Let $\{\kappa_i | i \in \underline{t}, \, t \leq r\}$ be the Kronecker invariants for the pair (A, B) of (2.1). Then, there exists a set of closed-loop generalized eigenvector $\{v_{ij} | i \in \underline{t}, j \in \underline{\kappa_i}\}$ (defined by (2.9)) satisfying the following equation.*

$$\operatorname{Im} Q(j) = \operatorname{Ker} \Lambda^{\kappa_1 - j - 1} V^{-1}, \quad (0 \leq j \leq \kappa_1 - 2), \tag{2.17}$$

where

$$V := [\, v_{11} \; v_{12} \; \cdots \; v_{1\kappa_1} \; v_{21} \; \cdots \; v_{2\kappa_2} \; \cdots \; v_{t\kappa_t} \,],$$

and

$$\Lambda := \mathit{block\ diag}\{J(\kappa_1), J(\kappa_2), \cdots, J(\kappa_t)\},$$

where

$$J(\kappa_i) := \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}_{\kappa_i \times \kappa_i}$$

Proof. Let $\mathcal{V}(j)$ be a subspace of $\mathbf{R}^n$ such that

$$\mathcal{V}(j) = \operatorname{Ker} \Lambda^{\kappa_1 - j - 1} V^{-1}.$$

Find, if it exists, the smallest positive integer σ satisfying

$$\kappa_{\sigma+1} \leq \kappa_1 - j - 1, \quad (\sigma < t)$$

and set $h = \sigma$. If such a σ does not exist, set $h = t$. From the structure of Jordan canonical form Λ, it follows that

$$\mathcal{V}(j) = \operatorname{Im} [\, v_{11}\, v_{12} \cdots v_{1\kappa_1 - j - 1} \cdots v_{h\kappa_1 - j - 1}\, v_{h+11} \cdots v_{t\kappa_t} \,].$$

Choosing vectors $p_{ij} \in \mathbf{R}^t (i \in \underline{t})$ according to the robust controller design algorithm of Section 2.4, $\mathcal{V}(j)$ is given by

$$\mathcal{V}(j) = \operatorname{Im} [\, V_1\, V_2 \cdots V_h\, V_{h+1} \cdots V_t \,], \tag{2.18}$$

where

$$V_i := [\, N_1 p_{i1}\;\, S_1 v_{i1} + N_1 p_{i2} \;\cdots\; S_1 v_{i\kappa_1 - j - 2} + N_1 p_{i\kappa_1 - j - 1} \,], \quad (\, 1 \leq i \leq h \,),$$

$$V_i := [\, N_1 p_{i1}\;\, S_1 v_{i1} + N_1 p_{i2} \;\cdots\; S_1 v_{i\kappa_i - 1} + N_1 p_{i\kappa_i} \,], \quad (\, h + 1 \leq i \leq t \,).$$

Since $\mathcal{V}(j)$ is invariant with respect to the selection p_{ij} $(\, i \in \underline{t},\ j = 2, 3, \ldots, \kappa_i \,)$, it is assumed with no loss of generality that $p_{ij} = 0$ $(\, i \in \underline{t},\ j = 2, 3, \ldots, \kappa_i \,)$. Rearranging the column vector of (2.18), it follows that

$$\mathcal{V}(j) = \operatorname{Im} [\, U_1\, U_2 \cdots U_{\kappa_t}\, U_{\kappa_t + 1} \cdots U_{\kappa_1 - j - 1} \,],$$

where

$$\begin{array}{lcll} U_i & := & S_1^{i-1} N_1 [\, p_{11}\, p_{21} \cdots p_{t1} \,], & (\, 1 \leq i \leq \kappa_t \,), \\ U_i & := & S_1^{\kappa_{t-1} - 1} N_1 [\, p_{11}\, p_{21} \cdots p_{(t-1)1} \,], & (\, \kappa_t + 1 \leq i \leq \kappa_{t-1} \,), \\ \vdots & & \vdots & \vdots \\ U_i & := & S_1^{\kappa_{h+1} - 1} N_1 [\, p_{11}\, p_{21} \cdots p_{(h+1)1} \,], & (\, \kappa_{h+2} + 1 \leq i \leq \kappa_{h+1} \,), \\ U_i & := & S_1^{\kappa_1 - j - 2} N_1 [\, p_{11}\, p_{21} \cdots p_{h1} \,], & (\, \kappa_{h+1} + 1 \leq i \leq \kappa_1 - j - 1 \,). \end{array}$$

From the fact that the matrix $[\, p_{11}\, p_{21} \cdots p_{t1} \,]$ can select all independent columns in $S_1^{i-1} N_1$ $(\, 1 \leq i \leq \kappa_t \,)$ and by continuing the same procedure, we have

$$\mathcal{V}(j) = \operatorname{Im} [\, N_1\, S_1 N_1 \cdots S_1^{\kappa_1 - j - 2} N_1 \,].$$

Obviously, the above construction generates a set of closed-loop generalized eigenvectors satisfying (2.17).

Proof of Theorem 2.1. From Lemma 2.1, using the set of closed-loop generalized eigenvectors $\{ v_{ij} | i \in \underline{t}, j \in \underline{\kappa_i} \}$ satisfying (2.17), then (2.15) in Theorem 2.1 is equivalent to

$$\tau_0 = \max \{\, j \mid \operatorname{Ker} \Lambda^{\kappa_1 - j - 1} V^{-1} \supset \operatorname{Im} E \,\}, \quad (0 \leq j \leq \kappa_1 - 2). \tag{2.19}$$

Suppose an integer τ_0 satisfies (2.19). Then, it follows that

$$\Lambda^{\kappa_1 - \tau_0 - 1} V^{-1} E = 0. \tag{2.20}$$

For the given Kronecker invariants $\{ \kappa_i | i \in \underline{t}, t \leq r \}$, we can always construct a constant feedback control gain F such that (2.9) are satisfied. Using this control gain F and the definitions of V and Λ in Lemma 2.1, (2.9) reduces to the following equation

$$(A + BF) V = V \Lambda,$$

or equivalently,

$$A + BF = V \Lambda V^{-1}. \tag{2.21}$$

With the aid of (2.21), the output $z(\kappa_1)$ is represented as follows :

$$\begin{array}{rcl} z(\kappa_1) & = & D x(\kappa_1) \\ & = & D\{ (A + BF)^{\kappa_1} x_0 + \sum_{i=0}^{\kappa_1 - 1} (A + BF)^i E s(\kappa_1 - i - 1) \} \\ & = & D\{ V \Lambda^{\kappa_1} V^{-1} x_0 + \sum_{i=0}^{\kappa_1 - 1} V \Lambda^i V^{-1} E s(\kappa_1 - i - 1) \}. \end{array} \tag{2.22}$$

It is easily shown that $\Lambda^{\kappa_1} \equiv 0$ and

$$\operatorname{Ker} \Lambda^{\kappa_1 - \tau_0 - 1} \subset \operatorname{Ker} \Lambda^{\kappa_1 - \tau_0} \subset \cdots \subset \operatorname{Ker} \Lambda^{\kappa_1 - 1}.$$

Then from (2.20), (2.22) is written as follows :

$$z(\kappa_1) = D\{ \sum_{i=1}^{\kappa_1 - \tau_0 - 2} V \Lambda^i V^{-1} E s(\kappa_1 - i - 1) + E s(\kappa_1 - 1) \}. \tag{2.23}$$

From (2.16),

$$\sum_{i=1}^{\kappa_1-\tau_0-2} V\Lambda^i V^{-1} Es(\kappa_1-i-1) = \sum_{i=1}^{\kappa_1-\tau_0-2}(A+BF)^i Es(\kappa_1-i-1)$$
$$\in \sum_{i=1}^{\kappa_1-\tau_0-2} A^i \operatorname{Im} E + \sum_{i=1}^{\kappa_1-\tau_0-2} A^{i-1}\mathcal{B}$$
$$= \mathcal{Y}_{\kappa_1-\tau_0-2}$$
$$\subset \operatorname{Ker} D. \qquad (2.24)$$

Therefore, we have $z(\kappa_1) = 0$ with the assumption (2.7). Next, for any integer $k > 0$, we have

$$\begin{aligned} z(\kappa_1+k) &= Dx(\kappa_1+k) \\ &= D\{(A+BF)^{\kappa_1+k}x_0 \\ &\quad + \sum_{i=\kappa_1+1}^{\kappa_1+k}(A+BF)^{i-1}Es(\kappa_1+k-i) \\ &\quad + \sum_{i=\kappa_1-\tau_0}^{\kappa_1}(A+BF)^{i-1}Es(\kappa_1+k-i) \\ &\quad + \sum_{i=2}^{\kappa_1-\tau_0-1}(A+BF)^{i-1}Es(\kappa_1+k-i) \\ &\quad + Es(\kappa_1+k-1)\} \\ &= 0, \end{aligned}$$

because $(A+BF)^{\kappa_1+k} = V\Lambda^{\kappa_1+k}V^{-1} = 0$ and by (2.7), (2.20) and (2.24). Then, after κ_1 steps which is the minimum state settling time, the constant control gain F can keep the output zero in any finite time interval in spite of input disturbances.

2.4 Design Algorithm of Robust Controller

Suppose now that the Kronecker invariants $\{\kappa_i | i \in \underline{t}, t \leq r\}$ have been selected. Also suppose that there exists an integer τ_0 such that (2.15) and (2.16) are satisfied. We can derive the following design algorithm for computing the constant feedback gain F which forces the output of the system (2.1) and (2.2) to zero in at most κ_1 time steps and thereafter keeps it zero for any initial state x_0 independent of the

disturbance input. In this algorithm, all the closed-loop eigenvalues are assigned to zero and the corresponding eigenvectors are selected to satisfy (2.17) with $j = \tau_0$.

Algorithm 2.1.

1. Compute the maximal rank matrices N, S that are the form of (2.11) and satisfy (2.12).

2. Compute the Kronecker invariants $\{\gamma_i | i \in \underline{t}\}$ of the pair (S_1, N_1).

   Rearrange $\gamma_i (i \in \underline{t})$ in decreasing order and set these coefficients as $\gamma(1), \gamma(2), \ldots, \gamma(t)$. Using $\gamma(i)$, put a vector p_{i1} as

   $$p_{i1} = e_{\gamma(i)},\ i \in \underline{t},$$

   where $e_{\gamma(i)}$ is i-th standard unit vector in $\mathbf{R}^t$.

3. Find, if it exists, the smallest positive integer g satisfying

   $$\kappa_{g+1} = 1,\ g < t,$$

   and set $\xi = g$. If it does not exist, set $\xi = t$.

4. Generate vector chains as follows :

   $$\begin{aligned} v_{ij} &= S_1 v_{ij-1} + N_1 p_{ij}, \\ w_{ij} &= S_2 v_{ij-1} + N_2 p_{ij}, \\ &\quad (i \in \underline{t}, j \in \underline{\kappa_i}), \end{aligned}$$

   where $v_{i0} = 0\,(i \in \underline{t})$ and $p_{ij}\,(i \in \underline{\xi}, j = 2, 3, \ldots, \kappa_i)$ are chosen arbitrarily.

5. For $i \in \underline{t}, j \in \underline{\kappa_i}$, compute $\overline{F}$ satisfying

   $$\overline{F} v_{ij} = w_{ij}.$$

   If no solution exists, alter one or more of the vector p_{ij}.

6. Find G such that $\overline{B} = BG$ and compute $F = G\overline{F}$.

2.5 Discussions

As we have mentioned in Section 2.1, there are some differences between the problem of robust output dead-beat control, which is considered here, and the problem of simultaneous disturbance localization and pole assignment or the problem of simultaneous dead-beat control and disturbance decoupling. It is possible for a system not to be solvable for the disturbance localization problem yet to be solvable for the asymptotic disturbance rejection problem. The robust output dead-beat controller has the property of asymptotic disturbance rejection. For the problem of simultaneous disturbance localization and pole assignment, the following theorems are obtained.

Theorem 2.2 (Chang and Rhodes [20]) *Given the system (2.1)-(2.3) and a symmetric set* $\overline{\Lambda}$ *of complex numbers, there exists an* F *such that* $< A + BF \mid \mathrm{Im}\, E > \subset \mathrm{Ker}\, D$ *and* $\sigma(A+BF) = \overline{\Lambda}$ *if and only if*

$$\mathrm{Im}\, E \subset \mathcal{R}^*, \tag{2.25}$$

and

$$< A \,|\, \mathcal{B} > = \mathcal{X},$$

where $\mathcal{R}^*$ *is the maximal* (A,B)*-controllability subspace contained in* $\mathrm{Ker}\, D$. *In this case,* F *may be chosen from Wonham [85]*

$$\underline{F(\mathcal{R}^*)} := \{F \mid (A+BF)\mathcal{R}^* \subset \mathcal{R}^*\}.$$

Theorem 2.3 (Wonham [85]) *Let* $\mathcal{V}^*$ *be the maximal* (A,B)*-invariant subspace contained in* $\mathrm{Ker}\, D$. *If* F *is chosen from*

$$\underline{F(\mathcal{V}^*)} := \{F \mid (A+BF)\mathcal{V}^* \subset \mathcal{V}^*\},$$

then

$$\mathcal{R}^* = < A + BF \mid \mathcal{B} \cap \mathcal{V}^* > . \tag{2.26}$$

Theorem 2.4 (Wonham [85])

$$\underline{F(\mathcal{V}^*)} \subset \underline{F(\mathcal{R}^*)}. \tag{2.27}$$

For the problem of dead-beat control, we can set $\overline{\Lambda} = 0$ in 2.2 with no loss of generality. On the other hand, for the problem of simultaneous disturbance localization and dead-beat control, we can derive the following corollary.

Corollary 2.1 *Under the assumptions of Theorem 2.1, there exists an output dead-beat controller with disturbance localization in the class of controllers that drive the output* $z(k)$ *from any* x_0 *to zero in* κ_1 *time steps if*

$$\overline{\mathcal{Y}}_{\kappa_1-2} \subset \mathrm{Ker}\, D\,, \tag{2.28}$$

where the subspace $\overline{\mathcal{Y}}_{\kappa_1-2}$ *is generated by the following sequence :*

$$\begin{aligned} \overline{\mathcal{Y}}_0 &= \mathrm{Im}\, E\,, \\ \overline{\mathcal{Y}}_1 &= A\overline{\mathcal{Y}}_0 + \mathcal{B} \cap \mathcal{V}\,, \\ \overline{\mathcal{Y}}_i &= A\overline{\mathcal{Y}}_{i-1} + \overline{\mathcal{Y}}_{i-1}\,, \\ & \quad (i = 2, 3, \ldots, \kappa_1 - j\,,\ 0 \le j \le \kappa_1 - 2)\,. \end{aligned} \tag{2.29}$$

where $\mathcal{V} := \mathrm{Im}\, V$,

$$V := [\, v_{11}\ v_{12} \cdots v_{1\kappa_1}\ v_{21} \cdots v_{2\kappa_2} \cdots v_{t\kappa_t} \,]\,,$$

and $v_{ij}\,(\, i \in \underline{t}\,, j \in \underline{\kappa_i}\,)$ *are defined by (2.9).*

From the results of Theorem 2.2 and Corollary 2.1, we have Theorem 2.5 for the problem of simultaneous disturbance localization and output dead-beat control, the formulation of which is severer than that of output dead-beat control with asymptotic disturbance rejection(Theorem 2.1).

Theorem 2.5 *Let the system pair* (A,B) *of (2.1) is reachable. Suppose that* $\mathrm{Im}\, E \subset \mathrm{Ker}\, D$ *and* $\mathcal{V} \subset \mathrm{Ker}\, D$ *where* $\mathcal{V}$ *is defined in Corollary 2.1. If there exists a solution for the problem of simultaneous disturbance localization and output dead-beat control, then*

$$\overline{\mathcal{Y}}_{\kappa_1-2} \subset \mathcal{R}^*\,, \tag{2.30}$$

where $\overline{\mathcal{Y}}_{\kappa_1-2}$ *is defined by Corollary 2.1 and* $\mathcal{R}^*$ *is defined by Theorem 2.2.*

Proof. From definitions of subspaces $\overline{\mathcal{Y}}_{\kappa_1-2}$ and , $\mathcal{R}^*$ and from Theorem 2.3, it follows that

$$\overline{\mathcal{Y}}_{\kappa_1-2} = \sum_{i=1}^{\kappa_1-2} A^i \mathrm{Im}\, E + \sum_{i=1}^{\kappa_1-2} A^{i-1}(\mathcal{B} \cap \mathcal{V})\,,$$

for $(A+BF)\mathcal{V} \subset \mathcal{V}$ and $\mathcal{V} \subset \operatorname{Ker} D$ and

$$\mathcal{R}^* = < A + BF_1 \mid \mathcal{B} \cap \mathcal{V}^* >,$$

for $F_1 \in \underline{F(\mathcal{V}^*)} \subset \underline{F(\mathcal{R}^*)}$. According to the solvability of Theorem 2.2, $\operatorname{Im} E \subset \mathcal{R}^*$ and as $\mathcal{R}^*$ is (A, B)-invariant, we have, for $i \in \underline{\kappa_1 - 2}$ and $F_1 \in \underline{F(\mathcal{V}^*)}$,

$$A^i \operatorname{Im} E \subset (A + BF_1)^i \operatorname{Im} E \subset (A + BF_1)^i \mathcal{R}^* \subset \mathcal{R}^* .$$

Next, from Corollary 2.1, $\overline{\mathcal{Y}}_{\kappa_1 - 2} \subset \operatorname{Ker} D$, and since $\mathcal{V}$ is assumed to be contained in $\operatorname{Ker} D$, $\mathcal{B} \cap \mathcal{V} \subset \mathcal{B} \cap \mathcal{V}^*$. Then, for $i = 0, 1, 2, \ldots, \kappa_1 - 3$ and $F_1 \in \underline{F(\mathcal{V}^*)}$,

$$A^i(\mathcal{B} \cap \mathcal{V}) \subset (A + BF_1)^i(\mathcal{B} \cap \mathcal{V}) \subset (A + BF_1)^i(\mathcal{B} \cap \mathcal{V}^*) \subset \mathcal{R}^* .$$

Therefore,

$$\overline{\mathcal{Y}}_{\kappa_1 - 2} = \sum_{i=1}^{\kappa_1 - 2} \{A^i \operatorname{Im} E \ + A^{i-1}(\mathcal{B} \cap \mathcal{V})\} \subset \mathcal{R}^* .$$

2.6 Numerical Examples

Example 2.1.

Consider the system

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} s(k), \tag{2.31}$$

$$z(k) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k), \tag{2.32}$$

for which, we can easily check that

1. The system (2.31) is reachable.

2. As $\operatorname{rank} B = 2 (< 3)$, it is obtained that

$$\overline{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} .$$

3. The assumption (2.7) is satisfied.

4. The Kronecker invariants are $\kappa_1 = \kappa_2 = 2$.

According to Algorithm 2.1, the maximal rank matrices N , S, which satisfy (2.12), are given by

$$N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ \hline -1 & 0 \\ 0 & -1 \end{array}\right], \tag{2.33}$$

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] . \tag{2.34}$$

Because $\kappa_1 = 2$ of the system (2.31), we can define the matrix $Q(0) := N_1$. From Theorem 2.1, there exists the nonnegative integer $\tau_0 = 0$ which satisfies the condition (2.15), and there also exists the subspace

$$\mathcal{Y}_{\kappa_1 - \tau_0 - 2} = \mathcal{Y}_0 = \operatorname{Im} E ,$$

which satisfies the condition (2.16). Therefore, it is concluded that there exists a robust output dead-beat controller for the system (2.31) and (2.32).

In order to obtain the constant feedback gain F, we put

$$p_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ p_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

because the Kronecker invariants of the pair (S_1, N_1) are $\gamma_1 = 2$, $\gamma_2 = 2$, and set $\xi = 2$. Generating vector chains, the equation $\overline{F} v_{ij} = w_{ij}$ $(i = 1, 2,\ j = 1, 2)$ is obtained by

$$\overline{F}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Thus, we obtain

$$\overline{F} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Finally, we get the constant feedback gain as the robust output dead-beat controller

$$F = G\overline{F} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where G satisfies $\overline{B} = BG$.

Example 2.2.

Consider the system

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} s(k), \tag{2.35}$$

$$z(k) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k), \tag{2.36}$$

for which, we can easily check that

1. The system (2.35) is reachable.
2. The assumption (2.7) is satisfied.
3. The Kronecker invariants are $\kappa_1 = 4$.

According to Algorithm 2.1, the maximal rank matrices N , S, which satisfy (2.12), are given by

$$N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ \hline -1 \end{array}\right], \tag{2.37}$$

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right]. \tag{2.38}$$

Because $\kappa_1 = 4$ of the system (2.35), we can define the matrices $Q(0)$, $Q(1)$, $Q(2)$. From Theorem 2.1, there exists the nonnegative integer $\tau_0 = 2$ which satisfies the condition (2.15), and there also exists the subspace $\mathcal{Y}_{\kappa_1-\tau_0-2} = \mathcal{Y}_0 = \operatorname{Im} E$, which satisfies the condition (2.16). Therefore, it is concluded that there exists a robust output dead-beat controller for the system (2.35) and (2.36). We can get the constant feedback gain as the robust dead-beat controller

$$F = \begin{bmatrix} 0 & -1 & 0 & 0 \end{bmatrix}.$$

On the other hand, the subspace $\overline{\mathcal{Y}}_{\kappa_1-2}$ which is defined by (2.29) is given by

$$\overline{\mathcal{Y}}_{\kappa_1-2} = \operatorname{Im}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}. \tag{2.39}$$

From Theorem 2.5, it is concluded that there does not exist a state feedback gain which gives a solution for the problem of simultaneous disturbance localization and output dead-beat control for the system (2.35)-(2.36).

2.7 Conclusion

In this chapter, we have considered the problem of output dead-beat control with asymptotic disturbance rejection by state feedback. The robustness of output dead-beat controllers with respect to the disturbance input is defined as asymptotic disturbance rejection, and a characterization of all such robust output dead-beat controllers is made. By using an eigenvalue-generalized eigenvector assignment, an algorithm is obtained for computing the constant state feedback gain which forces the output to zero in at most κ_1 time steps and keeps it zero for any initial state x_0 independent of disturbance input. The relation between the problem of output dead-beat control with asymptotic disturbance rejection and that of simultaneous disturbance localization and pole assignment has been considered. For the problem of output dead-beat control with asymptotically disturbance rejection, a broader characterization of such an output dead-beat controller is obtained than that which has already known.

Chapter 3

DEAD-BEAT CONTROLLERS BY DYNAMIC COMPENSATION

3.1 Introduction

The purpose of this chapter is to construct a robust dynamic output dead-beat controller by using an eigenvalue–generalized eigenvector assignment (Klein and Moore [42]). For the problem of pole assignment by dynamic compensation, Brasch and Pearson [18] show that, for the complete system, there exists a dynamic compensator with dimension of $(\nu_o - 1)$ which can arbitrarily assign all eigenvalues of the augmented system which is composed of the system to be controlled and the dynamic compensator, where ν_o is the observability index.

As in the case of state feedback systems, the choice of parameters of such dynamic compensators is, in general, not uniquely determined. That is, there exist some freedom beyond specification of the closed-loop eigenvalues. The robust controllers for discrete-time systems are adopted and some conditions are derived for the existence of a robust dynamic output dead-beat controller for a discrete-time linear time-invariant multivariable systems such that the finite output dead-beat regulation with asymptotic disturbance rejection occurs for arbitrary input disturbances. In this case, the conditions obtained are different from the problem of disturbance decoupling with stability(Wonham [85]), and it is possible for a system to exist an output dead-beat controller with asymptotic disturbance rejection even though there is no output dead-beat controller with disturbance decoupling.

The organization of this chapter is as follows. In Section 3.2, we describe some preliminary results and give the definitions. We give the solution for the problem of eigenvalue assignment using dynamic compensator without external disturbance input and an algorithm for computing the extended feedback controller using dynamic compensator of order $(\nu_o - 1)$ in Section 3.3. In Section 3.4, it is given the structure of a robust dynamic compensator with respect to the disturbance input. A characterization of all such robust dynamic dead-beat controllers is given in Section 3.5. A design algorithm is given in Section 3.6, which is a modified one obtained in Section 3.3, for computing the robust dynamic dead-beat controller which drives the output of the system to zero from any initial state in a minimum number of time steps and keeps it zero in the presence of disturbance input. Some numerical examples are given in Section 3.7.

3.2 Problem Statement and Preliminaries

Consider the time-invariant linear multivariable discrete-time system described by

$$x(k+1) = Ax(k) + Bu(k) + Es(k)\ ,\ \ x(0) = x_0\ , \tag{3.1}$$

$$y(k) = Cx(k)\ , \tag{3.2}$$

$$z(k) = Dx(k)\ ,\ \ k = 0, 1, 2, 3, \ldots, \tag{3.3}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $y \in \mathbf{R}^{m_y}$, $z \in \mathbf{R}^{m_z}$ and $s \in \mathbf{R}^{n_s}$ denote the state, control input, measurement output, regulated output and disturbance input, respectively. We assume that the triplet $(C\,,\ A\,,\ B)$ is *complete*. The control input $u(k)$ is synthesized by the following dynamic compensator :

$$w(k+1) = Ww(k) + Ky(k)\ , \tag{3.4}$$

$$u(k) = Hy(k) + Gw(k)\ , \tag{3.5}$$

where $w \in \mathbf{R}^{\ell}$ is the state of the dynamic compensator. Then, the augmented state space system derived from (3.1)-(3.5) is given by

$$\begin{bmatrix} x(k+1) \\ w(k+1) \end{bmatrix} = \begin{bmatrix} A+BHC & BG \\ KC & W \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} s(k) \tag{3.6}$$

$$y(k) = [C\ \ 0] \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}, \tag{3.7}$$

$$z(k) = [D\ \ 0] \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}. \tag{3.8}$$

Figure 3.1 illustrates the augmented closed-loop system (3.6)-(3.8). Let the observability index of $(C\,,\ A)$ be ν_o. For the problem of pole shifting by dynamic compensation with $\dim(W) = \nu_o - 1$, the following theorem is obtained by Brasch and Pearson [18].

Theorem 3.1 *Let $(C\,,\ A\,,\ B)$ be complete and let the observability index of $(C\,,\ A)$ be ν_o.*

Introduce the linear space $\mathcal{W}$, independent of $\mathcal{X}$, with $\dim(\mathcal{W}) = \nu_o - 1$. Then, for every symmetric set Λ^* of $n + \nu_o - 1$ complex numbers, there exist maps

$$G : \mathcal{W} \rightarrow \mathcal{U}\,,\ H : \mathcal{Y} \rightarrow \mathcal{U}\,,\ K : \mathcal{Y} \rightarrow \mathcal{W}\ and\ W : \mathcal{W} \rightarrow \mathcal{W}$$

such that (in a basis adopted to $\mathcal{X} \oplus \mathcal{W}$)

$$\sigma\left(\begin{bmatrix} A+BHC & BG \\ KC & W \end{bmatrix}\right) = \Lambda^*. \tag{3.9}$$

As in the case of state feedback system in Chapter 2, the choice of the quadruplet $(G\,,\ H\,,\ K\,,\ W)$ is, in general, not uniquely determined. That is, there exist some freedom beyond specification of the closed-loop eigenvalues of composite system (3.6). Using this flexibility to tune up the robustness of output dead-beat controllers, it is shown that under some specific conditions, such a dynamic controller can be constructed so as to resist the input disturbances. The problem to be treated in this chapter is the following.

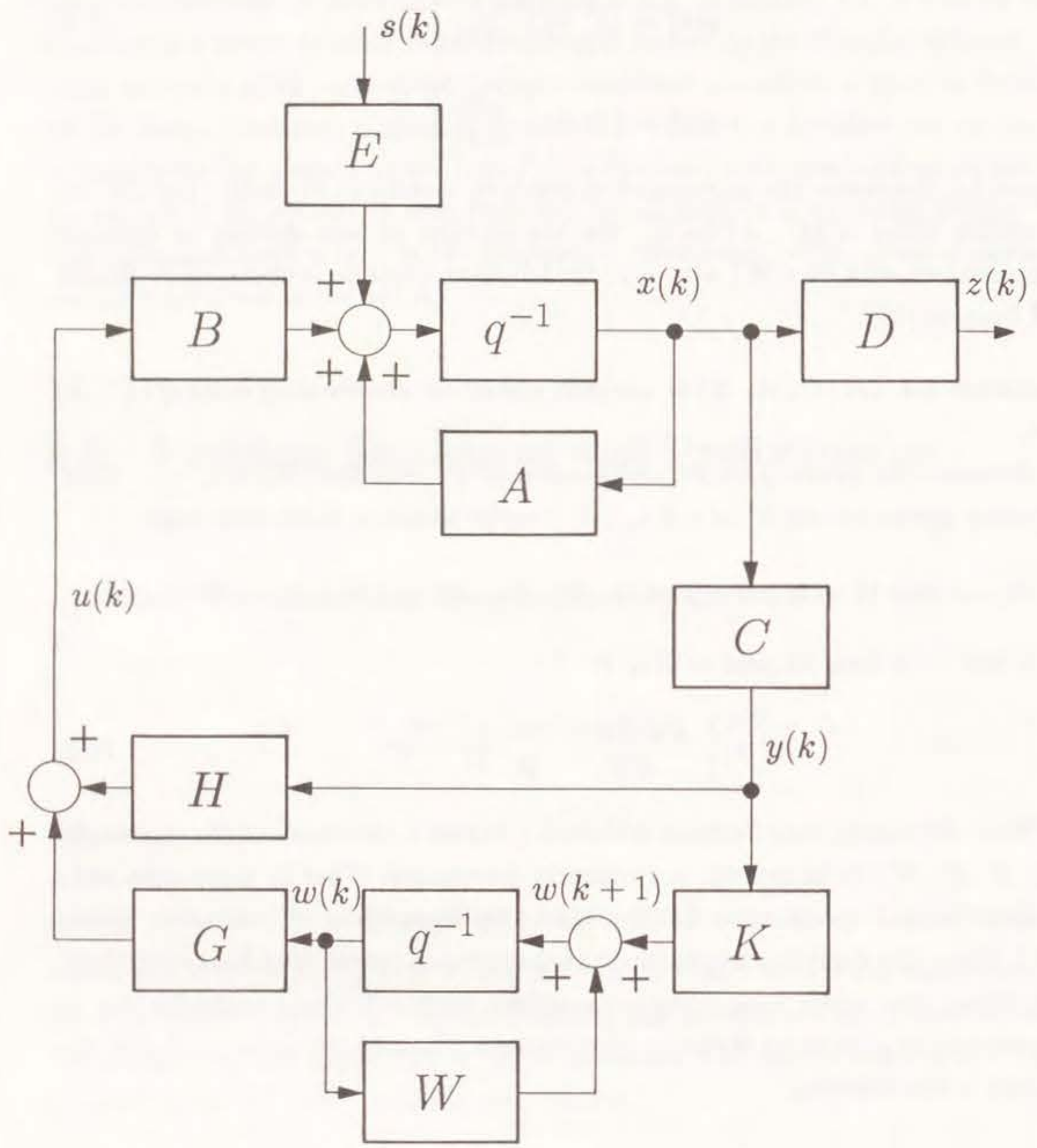

Figure 3.1: Block diagram of an augmented closed-loop control system.

[Robust Output Dead-Beat Control Problem by Dynamic Compensation]

For the system (3.1)-(3.3), construct the dynamic compensator

$$(G\,,\,H\,,\,K\,,\,W)$$

such that

$$z(T_f + k) = 0\,,\; k = 0, 1, 2, \ldots, \tag{3.10}$$

in which the number of steps T_f being minimum, in spite of the presence of disturbance input $s(k)$ $(0 \le k < \infty)$, where the number of steps T_f is the output settling time of the system (3.1)-(3.3).

If such a compensator $(\,G\,,\,H\,,\,K\,,\,W\,)$ exists, the controller is called to be a *robust dynamic output dead-beat controller* with respect to input disturbances. In order to formulate the problem, we need the following definitions.

Definition 3.1 *The dynamic compensator (3.4),(3.5) is said to be the strict output dead-beat controller if the regulated output $z(k)$ satisfies the following equation even if any external disturbances are introduced in arbitrary step :*

$$z(n^* + k) = 0\,,\; k = 0, 1, 2, \ldots, \tag{3.11}$$

where $n^* = n + \nu_o - 1$.

Definition 3.2 *The dynamic compensator (3.4),(3.5) is said to be the asymptotic output dead-beat controller if the regulated output*

$$z(k) \to 0 \;\; as \;\; k \to \infty\,,$$

even if any external disturbances are introduced in arbitrary step($< \infty$).

3.3 Eigenvalue Assignment by Dynamic Compensator

From Theorem 3.1, for the augmented system (3.6)-(3.8) without the external disturbance input, there exists a dynamic compensator of order $(\nu_o - 1)$ such that all eigenvalues of the augmented system are assigned arbitrarily. The extended output dead-beat controller using dynamic compensator of order $(\nu_o - 1)$, such that all eigenvalues of the augmented system are assigned to be zero (in order to obtain output dead-beat response) for the case without input disturbance, is constructed by the following algorithm(Wonham [85]).

Algorithm 3.1

1. Pick H_0 at random to make $A_0 := A + BH_0C$ cyclic, and set $A = A_0$.

2. Pick $b = Bu$ so as to make (A, b) reachable.

3. Set

$$W_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{\nu\times\nu}, g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\nu\times 1}, h^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times\nu} \tag{3.12}$$

where $\nu := \nu_o - 1$.

4. By eigenvalue assignment, compute $p^T : 1 \times n$, $\psi^T : 1 \times \nu$, so that

$$\begin{bmatrix} A & bh^T \\ 0 & W_0 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix} \begin{bmatrix} p^T & \psi^T \end{bmatrix}$$

has the desired spectrum Λ^*. For dead-beat control, choose $\Lambda^* = \{0\}$.

5. Compute $\hat{e}^T$, $\hat{e}_i^T : 1 \times m_y$ $(i \in \underline{\nu})$, not necessarily unique, such that

$$p^T = \hat{e}_1^T C + \hat{e}_2^T CA + \ldots + \hat{e}_\nu^T CA^{\nu-1} + \hat{e}^T CA^\nu. \tag{3.13}$$

6. Set $\tilde{A} := A + b\hat{e}^T C$ and compute $e_i^T : 1 \times m\,(i \in \underline{\nu})$ such that

$$p^T = e_1^T C + e_2^T C\tilde{A} + \ldots + e_\nu^T C\tilde{A}^{\nu-1} + \hat{e}^T C\tilde{A}^\nu, \tag{3.14}$$

where $\hat{e}^T$ in (3.14) is given by (3.13).

7. Set $T := W_0 + g\psi^T$, $k_i^T := h^T T^{i-1} K$ $(i \in \underline{\nu})$ and compute $k_i^T\,(i = \nu, \nu - 1, \ldots, 1)$ from

$$k_{\nu-i+1}^T = e_i^T + \theta_i \hat{e}^T + \theta_{i+1} k_1^T + \theta_{i+2} k_2^T + \ldots + \theta_\nu k_{\nu-i}^T, \tag{3.15}$$

where $\theta_i \in \mathbf{R}$ are the coefficients of the m.p. of T.

8. Compute r_i from

$$r_1^T = \hat{e}^T C, \; r_{i+1}^T = r_i^T \tilde{A} + k_i^T C, \;\; (i \in \underline{\nu - 1}). \tag{3.16}$$

9. Compute

$$\begin{bmatrix} K & R \end{bmatrix} = \begin{bmatrix} h^T \\ h^T T \\ \vdots \\ h^T T^{\nu-1} \end{bmatrix}^{-1} \begin{bmatrix} k_1^T & r_1^T \\ k_2^T & r_2^T \\ \vdots & \vdots \\ k_\nu^T & r_\nu^T \end{bmatrix},$$

$$W = T - Rbh^T, \; G = uh^T, \; H = u\hat{e}^T.$$

As almost all systems satisfy $n < m\nu_o$, (3.14) does not give a unique set $\{\hat{e}^T, e_i^T\ (i \in \underline{\nu})\}$, that is, Algorithm 3.1 above generates the class of dynamic compensators which drive the system state to zero in $n^* (= n + \nu_o - 1)$ steps in the case where any input disturbance does not exist.

The purpose of this chapter is to find the dynamic compensator which drives the output to zero in n^* steps even if any input disturbances are introduced in arbitrary step from the class of the dynamic compensators constructed by Algorithm 3.1. By Definition 3.1, such a dynamic compensator is called *the strict output dead-beat controller.*

In the case of

$$s(k) \neq 0 \quad \text{for} \quad 0 \le k \le T_d,$$
$$s(k) = 0 \quad \text{for} \quad T_d < k\ (T_d < \infty),$$

that is, the system is in operation under external disturbances when $0 \le k \le T_d\ (T_d < \infty)$, it is sufficient to consider a less restrictive output dead-beat controller than strict output dead-beat controller. It is called *the asymptotic output dead-beat controller* by Definition 3.2.

3.4 Robust Dynamic Compensator

Compute maximal rank matrices

$$N_e = \begin{bmatrix} N_{e1} \\ N_{e2} \end{bmatrix}, S_e = \begin{bmatrix} S_{e1} \\ S_{e2} \end{bmatrix}, \tag{3.17}$$

satisfying

$$\left[\begin{array}{cc|c} A & bh^T & 0 \\ 0 & W_0 & g \end{array}\right] \begin{bmatrix} N_{e1} \\ N_{e2} \end{bmatrix} = 0,$$

$$\left[\begin{array}{cc|c} A & bh^T & 0 \\ 0 & W_0 & g \end{array}\right] \begin{bmatrix} S_{e1} \\ S_{e2} \end{bmatrix} = I_{n+\nu},$$

where A, b, g, h^T, W_0 are obtained from Algorithm 3.1 and $\nu = \nu_o - 1$. Using (3.17), we define the matrix $Q_e(j)$ as follows :

$$Q_e(j) := [\, N_{e1}\; S_{e1}N_{e1}\; \ldots\; S_{e1}^{n^*-j-2}N_{e1} \,],\ (0 \le j \le n^* - 2). \tag{3.18}$$

Further, we define vectors $\tilde{e}^T$, $\tilde{e}_i^T$: $1 \times m_y\, (i \in \underline{\nu})$, not necessarily unique, satisfying

$$p^T = \tilde{e}_1^T C + \tilde{e}_2^T CA + \ldots + \tilde{e}_\nu^T CA^{\nu-1} + \tilde{e}^T CA^\nu. \tag{3.19}$$

where p^T is obtained from Algorithm 3.1. By using $Q_e(j)$ defined in (3.18) and the class of vectors $\{\, \tilde{e}^T, \tilde{e}_i^T\, (i \in \underline{\nu})\, \}$ satisfying (3.19), we obtain Theorem 3.2 with respect to state dead-beat control of (3.6).

Theorem 3.2 *Let $(\, C\, ,\, A\, ,\, B\,)$ be complete and let the observability index of $(\, C\, ,\, A\,)$ be ν_o. Let k_0 be the integer such that $0 \le k_0 \le n^* - 2\, (\, n^* := n + \nu_o - 1\,)$. Then, there exists a dynamic compensator using which the augmented state $[\, x(\cdot)^T,\ w(\cdot)^T\,]^T$ is driven to zero in n^* steps despite of external disturbances introduced within k_0 step, if there exists a set of vectors $\{\, \tilde{e}^T, \tilde{e}_i^T\, (i \in \underline{\nu})\, \}\, (\, \nu := \nu_o - 1\,)$ satisfying the following equation in the class of the vector set satisfying (3.19),*

$$\operatorname{Im} Q_e(k_0) \supset \begin{bmatrix} I \\ \tilde{R} \end{bmatrix} \operatorname{Im} E, \tag{3.20}$$

where

$$\tilde{R} := \begin{bmatrix} \tilde{r}_1 & \tilde{r}_2 & \cdots & \tilde{r}_\nu \end{bmatrix}^T,$$

$$\tilde{r}_1^T = \tilde{e}^T C,\ \ \tilde{r}_{i+1}^T = \tilde{r}_i^T A + \tilde{k}_i^T C,\ \ (i \in \underline{\nu - 1}),$$

$$\tilde{k}_{\nu-j+1}^T = \tilde{e}_j^T + \theta_j \tilde{e}^T + \theta_{j+1}\tilde{k}_1^T + \theta_{j+2}\tilde{k}_2^T + \cdots + \theta_\nu \tilde{k}_{\nu-j}^T,\ (\, j = \nu, \nu-1, \ldots, 1\,),$$

where $\theta_i \in \mathbf{R}$ are the coefficients of the m.p. of T $(:= W_0 + g\psi^T)$, which is defined in Algorithm 3.1.

To prove Theorem 3.2, we need two preliminary results.

Lemma 3.1 *Consider the set of vectors $\{\, \tilde{e}^T, \tilde{e}_i^T\, (i \in \underline{\nu})\, \}$ satisfying (3.19). Compute $\bar{e}^T, \bar{e}_i^T\, (i \in \underline{\nu})$ such that*

$$\begin{aligned} \bar{e}^T &= \tilde{e}^T, \\ \bar{e}_i^T &= \tilde{e}_i^T - \tilde{e}_{i+1}^T CA_1 - \cdots - \tilde{e}_\nu^T CA_{\nu-i} - \tilde{e}^T CA_{\nu-i+1}, \end{aligned} \tag{3.21}$$

where $A_i := A^{i-1}b\tilde{e}^T = A^{i-1}b\bar{e}^T$. Then, the set of vectors $\{\bar{e}^T, \bar{e}_i^T\, (i \in \underline{\nu})\}$ satisfies the following equation

$$p^T = \bar{e}_1^T C + \bar{e}_2^T C\tilde{A} + \ldots + \bar{e}_\nu^T C\tilde{A}^{\nu-1} + \bar{e}^T C\tilde{A}^\nu \tag{3.22}$$

where p^T is the same vector in (3.19) and $\tilde{A} := A + b\tilde{e}^T C = A + b\bar{e}^T C$.

The proof of Lemma 3.1 is straightforward by algebraic computation and is omitted.

Lemma 3.2 *Under the construction of Lemma 3.1, consider the vectors*

$$r_i^T,\ \tilde{r}_i^T,\ \ (i \in \underline{\nu})$$

defined by

$$k_{\nu-j+1}^T = \bar{e}_j^T + \theta_j \bar{e}^T + \theta_{j+1}k_1^T + \theta_{j+2}k_2^T + \cdots + \theta_\nu k_{\nu-j}^T,\ (\, j = \nu, \nu-1, \ldots, 1\,).$$

$$r_1^T = \bar{e}^T C,\ r_{i+1}^T = r_i^T \tilde{A} + k_i^T C,\ \ (i \in \underline{\nu - 1}),$$

$$\tilde{k}_{\nu-j+1}^T = \tilde{e}_j^T + \theta_j \tilde{e}^T + \theta_{j+1}\tilde{k}_1^T + \theta_{j+2}\tilde{k}_2^T + \cdots + \theta_\nu \tilde{k}_{\nu-j}^T,\ (\, j = \nu, \nu-1, \ldots, 1\,),$$

$$\tilde{r}_1^T = \tilde{e}^T C,\ \tilde{r}_{i+1}^T = \tilde{r}_i^T A + \tilde{k}_i^T C,\ \ (i \in \underline{\nu - 1}),$$

where $\theta_i \in \mathbf{R}$ are the coefficients of the m.p. of $T\, (:= W_0 + g\psi\,)$, which is defined in Algorithm 3.1. Then, for each $i \in \underline{\nu}$,

$$r_i^T = \tilde{r}_i^T. \tag{3.23}$$

Proof. For $i = 1$, because of $\bar{e}^T = \tilde{e}^T$ (Lemma 3.1), it is clear that

$$r_1^T = \bar{e}^T C = \tilde{e}^T C = \tilde{r}_1^T ,$$

and

$$\begin{aligned} k_1^T &= \bar{e}_\nu^T + \theta_\nu \bar{e}^T \\ &= (\tilde{e}_\nu^T - \tilde{e}^T C A_1) + \theta_\nu \tilde{e}^T \\ &= (\tilde{e}_\nu^T + \theta_\nu \tilde{e}^T) - \bar{e}_\nu^T C b \bar{e}^T \\ &= \tilde{k}_1^T - r_1^T b \bar{e}^T . \end{aligned}$$

For $i = 2$, from the above definition and Lemma 3.1,

$$\begin{aligned} r_2^T &= r_1^T \tilde{A} + k_1^T C \\ &= r_1^T (A + b\bar{e}^T C) + (\bar{e}_\nu^T + \theta_\nu \bar{e}^T) C \\ &= r_1^T A + (\bar{e}_\nu^T + \bar{e}^T C A_1) C + \theta_\nu \bar{e}^T C \\ &= r_1^T A + (\tilde{e}_\nu^T + \theta_\nu \tilde{e}^T) C \\ &= \tilde{r}_1^T A + \tilde{k}_1^T C \\ &= \tilde{r}_2^T , \end{aligned}$$

and

$$\begin{aligned} k_2^T &= \bar{e}_{\nu-1}^T + \theta_{\nu-1} \bar{e}^T + \theta_\nu k_1^T \\ &= \tilde{e}_{\nu-1}^T - \tilde{e}_\nu^T C A_1 - \tilde{e}^T C A_2 + \theta_{\nu-1} \tilde{e}^T \\ &\quad + \theta_\nu \{(\tilde{e}_\nu^T - \tilde{e}^T C A_1) + \theta_\nu \tilde{e}^T\} \\ &= \tilde{e}_{\nu-1}^T + \theta_{\nu-1} \tilde{e}^T + \theta_\nu (\tilde{e}_\nu^T + \theta_\nu \tilde{e}^T) \\ &\quad - \tilde{e}_\nu^T C A_1 - \tilde{e}^T C A_2 - \theta_\nu \tilde{e}^T C A_1 \\ &= \tilde{k}_2^T - (\tilde{k}_1^T C + \tilde{r}_1^T A) b \bar{e}^T \\ &= \tilde{k}_2^T - \tilde{r}_2^T b \bar{e}^T \\ &= \tilde{k}_2^T - \tilde{r}_2^T b \bar{e}^T . \end{aligned}$$

Next, we assume that (3.23) is true for $i = h\,(> 0)$, that is

$$r_h^T = \tilde{r}_h^T \ \text{ and } \ k_h^T = \tilde{k}_h^T - r_h^T b \bar{e}^T .$$

Then, from the definition of r_{h+1}^T,

$$\begin{aligned} r_{h+1}^T &= r_h^T \tilde{A} + k_h^T C &&= r_h^T A + r_h^T b \bar{e}^T C + k_h^T C \\ &= \tilde{r}_h^T A + \tilde{k}_h^T C &&= \tilde{r}_{h+1}^T . \end{aligned}$$

Thus, for $i \in \underline{\nu}$, we obtained the identity (3.23).

Lemma 3.3 *Under the construction of Lemma 3.1 and 3.2, the following equation holds*

$$\psi^T R + p^T = r_\nu^T \tilde{A} + k_\nu^T C , \tag{3.24}$$

where ψ^T is given by Algorithm 3.1 and

$$R := \begin{bmatrix} r_1 & r_2 & \cdots & r_\nu \end{bmatrix}^T .$$

Proof. As $T := W_0 + g\psi^T$ is cyclic and m.p.of T is

$$\alpha(\lambda) = \lambda^\nu - \theta_\nu \lambda^{\nu-1} - \theta_{\nu-1} \lambda^{\nu-2} - \cdots - \theta_2 \lambda - \theta_1 ,$$

it is obtained that $\psi^T = [\theta_1\ \theta_2\ \ldots\ \theta_\nu]$, and $\psi^T R + p^T = \sum_{i=1}^{\nu} \theta_i r_i + p^T$, where, for $i \in \underline{\nu}$,

$$\theta_i r_i^T = \theta_i \bar{e}^T C \tilde{A}^{i-1} + \theta_i (k_1^T C \tilde{A}^{i-2} + k_2^T C \tilde{A}^{i-3} + \cdots + k_{i-1}^T C) ,$$

and p^T is given by (3.22). Then, we have

$$\begin{aligned} \psi^T R + p^T &= \textstyle\sum_{i=1}^{\nu} \theta_i r_i + p^T \\ &= \textstyle\sum_{i=1}^{\nu} \{\theta_i \bar{e}^T C \tilde{A}^{i-1} \\ &\quad + \theta_i (k_1^T C \tilde{A}^{i-2} + k_2^T C \tilde{A}^{i-3} + \cdots + k_{i-1}^T C)\} \\ &\quad + \bar{e}_1^T C + \bar{e}_2^T C \tilde{A} + \ldots + \bar{e}_\nu^T C \tilde{A}^{\nu-1} + \bar{e}^T C \tilde{A}^\nu \\ &= ((\cdots (\bar{e}^T C \tilde{A} + (\bar{e}_\nu^T + \theta_\nu \bar{e}^T) C) \tilde{A} \\ &\quad + (\bar{e}_{\nu-1}^T + \theta_{\nu-1} \bar{e}^T + \theta_\nu k_1) C) \tilde{A} \\ &\quad + \cdots) \tilde{A} + (\bar{e}_1^T + \theta_1 \bar{e}^T + \theta_2 k_1^T + \cdots + \theta_\nu k_{\nu-1}^T) C \\ &= ((\cdots (r_1^T \tilde{A} + k_1^T C) \tilde{A} + k_2^T C) \tilde{A} + k_3^T C) \tilde{A} \\ &\quad + k_4^T C) \tilde{A} + \cdots) \tilde{A} + k_\nu^T C \\ &= r_\nu^T \tilde{A} + k_\nu^T C . \end{aligned}$$

Lemma 3.4

$$\psi^T \tilde{R} + p^T = \tilde{r}_\nu^T A + \tilde{k}_\nu^T C . \tag{3.25}$$

Proof. From Lemma 3.2, $R = \tilde{R}$. Then, $\psi^T R + p^T = \psi^T \tilde{R} + p^T$. On the other hand,

$$\begin{aligned} & r_\nu^T \tilde{A} + k_\nu^T C \\ &\qquad = r_\nu^T A + r_\nu^T b \bar{e}^T C + k_\nu^T C \\ &\qquad = \tilde{r}_\nu^T A + (k_\nu^T + r_\nu^T b \bar{e}^T) C \\ &\qquad = \tilde{r}_\nu^T A + \tilde{k}_\nu^T C . \end{aligned}$$

From the identity of (3.24), (3.25) holds.

Proof of Theorem 3.2. Using the same notations defined in Algorithm 3.1, set

$$A_0 := \begin{bmatrix} A & bh^T \\ 0 & W_0 \end{bmatrix}, \ B_0 := \begin{bmatrix} 0 \\ g \end{bmatrix}, \ F_0 := \begin{bmatrix} p^T & \psi^T \end{bmatrix} .$$

From Lemma 3.1, 3.2, 3.3 and 3.4, it follows that

$$[A_0 + B_0 F_0] \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix} \begin{bmatrix} A + BHC & BG \\ KC & W \end{bmatrix} . \tag{3.26}$$

To achieve the augmented state dead-beat control in n^* steps, all eigenvalues of the system matrix $A_0 + B_0 F_0$ is assigned to zero, so we have the following (real) Jordan canonical structure :

$$A_0 + B_0 F_0 = V_0 \Lambda_0 V_0^{-1} , \tag{3.27}$$

where V_0 is the matrix of generalized closed-loop eigenvectors and

$$\Lambda_0 := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n^* \times n^*}$$

From the structure of $Q_e(j)$, it follows that

$$\operatorname{Im} Q_e(k_0) = \operatorname{Ker} \Lambda_0^{n^* - k_0 - 1} V_0^{-1} , \quad \text{for } 0 \leq k_0 \leq n^* - 2 . \tag{3.28}$$

Then, (3.20) is equivalent to

$$\operatorname{Ker} \Lambda_0^{n^* - k_0 - 1} V_0^{-1} \supset \begin{bmatrix} I \\ \tilde{R} \end{bmatrix} \operatorname{Im} E . \tag{3.29}$$

Suppose a set of vectors

$$\{ \tilde{e}^T , \tilde{e}_i^T \, (i \in \underline{\nu}) \}$$

(in the class of vector set given by (3.19)) satisfies (3.20) which is equal to (3.29). At n^*-th step, using the equalities (3.26) and (3.27), the extended state $[\, x(n^*)^T \; w(n^*)^T \,]^T$ of the augmented system (3.6) with arbitrary disturbance input for $0 \leq k \leq k_0$ $(k_0 \leq n^* - 2)$ is as follows.

$$\begin{aligned} \begin{bmatrix} x(n^*) \\ w(n^*) \end{bmatrix} = & \begin{bmatrix} A + BHC & BG \\ KC & W \end{bmatrix}^{n^*} \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} \\ & + \textstyle\sum_{i=n^*-k_0-1}^{n^*-1} \begin{bmatrix} A + BHC & BG \\ KC & W \end{bmatrix}^i \begin{bmatrix} E \\ 0 \end{bmatrix} s(n^* - i - 1) \\ = & \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix}^{-1} (A_0 + B_0 F_0)^{n^*} \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix} \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} \\ & + \textstyle\sum_{i=n^*-k_0-1}^{n^*-1} \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix}^{-1} (A_0 + B_0 F_0)^i \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix} \\ & \qquad \times \begin{bmatrix} E \\ 0 \end{bmatrix} s(n^* - i - 1) \\ = & \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix}^{-1} V_0 \Lambda^{n^*} V_0^{-1} \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix} \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} \\ & + \textstyle\sum_{i=n^*-k_0-1}^{n^*-1} \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix}^{-1} V_0 \Lambda^i V_0^{-1} \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix} \\ & \qquad \times \begin{bmatrix} E \\ 0 \end{bmatrix} s(n^* - i - 1) . \end{aligned}$$

It is easily shown that $\Lambda^{n^*} \equiv 0$ and

$$\operatorname{Ker} \Lambda^{n^* - k_0 - 1} V_0^{-1} \subset \operatorname{Ker} \Lambda^{n^* - k_0} V_0^{-1} \subset \cdots \operatorname{Ker} \Lambda^{n^* - 1} .$$

And from (3.29), we have

$$\Lambda^{n^* - k_0 - 1} V_0^{-1} \begin{bmatrix} I \\ \tilde{R} \end{bmatrix} \operatorname{Im} E = \Lambda^{n^* - k_0 - 1} V_0^{-1} \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix} \operatorname{Im} \begin{bmatrix} E \\ 0 \end{bmatrix} = 0 .$$

Using these relations, we obtain

$$\begin{bmatrix} x(n^*) \\ w(n^*) \end{bmatrix} = 0 .$$

And for any integer $k \, (> 0)$, it is also shown that

$$\begin{bmatrix} x(n^* + k) \\ w(n^* + k) \end{bmatrix} = 0 .$$

Then, the proof is finished.

3.5 Robust Output Dead-Beat Controllers

Theorem 3.2 states that for some finite step $T_d(<\infty)$, if (3.20) is satisfied by $k_0 = T_d$, there exists a dynamic dead-beat controller (G, H, K, W) in weak sense. For the problem of existence of an *asymptotic output dead-beat controller*, the following theorem is obtained.

Theorem 3.3 *The dynamic compensator*

$$(G, H, K, W),$$

which satisfies Theorem 3.2, is an asymptotic output dead-beat controller.

In Theorem 3.3, in the case of $n^*-1 < T_d$, the quadruple (G, H, K, W) which satisfies Theorem 3.2 can not drive the output $z(\cdot)$ to zero in n^* steps and can not keep $z(\cdot)$ zero after n^*-th step. Hence, such a dynamic compensator (G, H, K, W) is not a *strict output dead-beat controller.*

For a characterization of the class of strict output dead-beat controllers, we define the subspace sequence $\mathcal{Z}_\mu$ as

$$\begin{aligned} \mathcal{Z}_0 &= \begin{bmatrix} I \\ \tilde{R} \end{bmatrix} \operatorname{Im} E, \\ \mathcal{Z}_1 &= A_0\mathcal{Z}_0 + \operatorname{Im} B_0, \\ \mathcal{Z}_i &= A_0\mathcal{Z}_{i-1} + \mathcal{Z}_{i-1}, \\ & \quad (i = 2, 3, \ldots, n^* - j,\ 0 \le j \le n^* - 2), \end{aligned} \tag{3.30}$$

where A_0, B_0 are defined in the proof of Theorem 3.2 and $\tilde{R}$ is defined in Theorem 3.2. By using (3.18) and (3.30), we obtain Theorem 3.4 for the existence of a *strict output dead-beat controller.*

Theorem 3.4 *Let (C, A, B) be complete and let the observability index of (C, A) be ν_o. Assume that $\operatorname{Im} E \subset \operatorname{Ker} D$. There exists a (robust) strict output dead-beat controller if the following two conditions are satisfied.*

1. *There exists a nonnegative integer τ^* satisfying the following condition :*

$$\tau^* = \max\{\, j \mid \operatorname{Im} Q_c(j) \supset \mathcal{Z}_0,\ 0 \le j \le n^* - 2\,\}. \tag{3.31}$$

2. *For the integer τ^* which satisfies (3.31), the subspace $\mathcal{Z}_{n^*-\tau^*-2}$, which is defined by (3.30), satisfies the following :*

$$\mathcal{Z}_{n^*-\tau^*-2} \subset \operatorname{Ker} \begin{bmatrix} D & 0 \end{bmatrix}. \tag{3.32}$$

Proof. From (3.28), (3.31) is equivalent to

$$\tau^* = \max\{\, j \mid \operatorname{Ker} \Lambda^{n^*-j-1} V_0^{-1} \supset \mathcal{Z}_0,\ 0 \le j \le n^* - 2\,\}. \tag{3.33}$$

Suppose that there exists an integer τ^* which satisfies (3.33). That is

$$\Lambda^{n^*-\tau^*-1} V_0^{-1} \begin{bmatrix} I & 0 \\ \tilde{R} & I \end{bmatrix} \begin{bmatrix} E \\ 0 \end{bmatrix} = 0. \tag{3.34}$$

Then, using (3.34), in the same way of the proof of Theorem 3.2, it is shown that

$$z(n^* + k) = 0 \quad \text{for} \quad k \ge 0$$

and so the proof is finished.

Remark. For the problem of simultaneous disturbance localization and dead-beat control, we can derive the following corollary.

Corollary 3.1 *Under the assumption of Theorem 3.4, there exists a dynamic output dead-beat controller with disturbance localization in the class of controllers which bring the output of the system from any initial state to zero in at most n^* time steps and thereafter keeps the output zero, if*

$$\bar{\mathcal{Z}}_{n^*-2} \subset \operatorname{Ker} \begin{bmatrix} D & 0 \end{bmatrix}, \tag{3.35}$$

where the subspace $\bar{\mathcal{Z}}_{n^*-2}$ is generated by the following sequence :

$$\begin{aligned} \bar{\mathcal{Z}}_0 &= \begin{bmatrix} I \\ \tilde{R} \end{bmatrix} \operatorname{Im} E, \\ \bar{\mathcal{Z}}_1 &= A_0\bar{\mathcal{Z}}_0 + \operatorname{Im} \mathcal{B}_0 \cap \mathcal{V}_0, \\ \bar{\mathcal{Z}}_i &= A_0\bar{\mathcal{Z}}_{i-1} + \bar{\mathcal{Z}}_{i-1}, \\ & \quad (i = 2, 3, \ldots, n^* - j,\ 0 \le j \le n^* - 2), \end{aligned}$$

in which $\mathcal{V}_0 = \operatorname{Im} V_0$ and V_0 is defined by (3.27).

3.6 Design Algorithm of Controllers

Suppose now that there exists a *strict output dead-beat controller* which satisfies Theorem 3.4. By modifying Algorithm 3.1, we derive a design algorithm for computing the quadruple (G, H, K, W) of the strict output dead-beat controller that brings the output from any initial state to zero in at most $n^*(= n + \nu_0 - 1)$ time steps and thereafter keeps the output zero in spite of the existence of unknown input. In this algorithm, all closed-loop eigenvalues of the augmented system are assigned zero and the corresponding eigenvectors are selected to satisfy (3.29) with $k_0 = r^*$.

Algorithm 3.2.

1. Pick H_0 at random to make $A_0 := A + BH_0C$ cyclic, and set $A = A_0$.

2. Pick $b = Bu$ so as to make (A, b) reachable.

3. Set

$$W_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{\nu\times\nu}, g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\nu\times 1}, h^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times\nu},$$

where $\nu := \nu_o - 1$.

4. By eigenvalue assignment, compute $p^T : 1 \times n$, $\psi^T : 1 \times \nu$, so that

$$\begin{bmatrix} A & bh^T \\ 0 & W_0 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix} \begin{bmatrix} p^T & \psi^T \end{bmatrix}$$

has the desired spectrum Λ^*. For dead-beat control, choose $\Lambda^* = \{0\}$.

5. Compute $\tilde{e}^T$, $\tilde{e}_i^T : 1 \times m_y$ $(i \in \underline{\nu})$, not necessarily unique, such that

$$p^T = \tilde{e}_1^T C + \tilde{e}_2^T CA + \ldots + \tilde{e}_\nu^T CA^{\nu-1} + \tilde{e}^T CA^\nu .$$

6. Compute $\vec{e}^T, \vec{e}_i^T : 1 \times m\,(i \in \underline{\nu})$ such that

$$\begin{aligned} \vec{e}^T &= \tilde{e}^T, \\ \vec{e}_i^T &= \tilde{e}_i^T - \tilde{e}_{i+1}^T CA_1 - \cdots - \tilde{e}_\nu^T CA_{\nu-i} - \tilde{e}^T CA_{\nu-i+1}, \end{aligned}$$

where $A_i := A^{i-1}b\tilde{e}^T = A^{i-1}b\vec{e}^T$.

7. Set $T := W_0 + g\psi^T$, and compute k_i^T, $\tilde{k}^T$ $(i = \nu, \nu-1, \ldots, 1)$ from

$$k_{\nu-i+1}^T = \vec{e}_i^T + \theta_i\vec{e}^T + \theta_{i+1}k_1^T + \theta_{i+2}k_2^T + \ldots + \theta_\nu k_{\nu-i}^T,$$

$$\tilde{k}_{\nu-i+1}^T = \tilde{e}_i^T + \theta_i\tilde{e}^T + \theta_{i+1}\tilde{k}_1^T + \theta_{i+2}\tilde{k}_2^T + \ldots + \theta_\nu \tilde{k}_{\nu-i}^T,$$

where $\theta_i \in \mathbf{R}$ are the coefficients of the minimal polynomial of T.

8. Compute $\tilde{r}_i$ from

$$\tilde{r}_1^T = \tilde{e}^T C, \ \ \tilde{r}_{i+1}^T = \tilde{r}_i^T A + \tilde{k}_i^T C, \quad (i \in \underline{\nu - 1}). \tag{3.36}$$

9. Compute

$$\begin{bmatrix} K & \tilde{R} \end{bmatrix} = \begin{bmatrix} h^T \\ h^T T \\ \vdots \\ h^T T^{\nu-1} \end{bmatrix}^{-1} \begin{bmatrix} k_1^T & \tilde{r}_1^T \\ k_2^T & \tilde{r}_2^T \\ \vdots & \vdots \\ k_\nu^T & \tilde{r}_\nu^T \end{bmatrix},$$

$$W = T - \tilde{R}bh^T, \ G = uh^T, \ H = u\vec{e}^T.$$

3.7 Numerical Examples

Example 3.1.

Consider the complete system

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} s(k), \tag{3.37}$$

$$y(k) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(k), \tag{3.38}$$

$$z(k) = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix} x(k), \tag{3.39}$$

with the observability index $\nu_o = 4$. From Theorem 3.1, we can construct the dynamic compensator of order $3 (= \nu_o - 1)$, by which all closed-loop eigenvalues of the augmented system are assigned to zero. Using Algorithm 3.2, the dynamic compensator is obtained as follows :

$$w(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} w(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} y(k), \tag{3.40}$$

$$u(k) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} y(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} w(k). \tag{3.41}$$

For the system (3.37)–(3.39), it is shown that $\tau^* = 5$ satisfies the conditions (3.31) and (3.32) of Theorem 3.4. Therefore, the dynamic compensator (3.40) and (3.41) is a strict output dead-beat controller for the system (3.37)–(3.39).

Example 3.2.

Consider the complete system

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s(k), \tag{3.42}$$

$$y(k) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(k), \tag{3.43}$$

$$z(k) = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix} x(k), \tag{3.44}$$

with the observability index $\nu_o = 4$.

As Example 3.1, we can obtain the dynamic compensator for pole shifting requirement,

$$w(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} w(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} y(k), \tag{3.45}$$

$$u(k) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} y(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} w(k). \tag{3.46}$$

For the system (3.42)–(3.44), it is easily shown that there is no τ^* which satisfies (3.31) and (3.32) of Theorem 3.4. Therefore, the dynamic compensator (3.45)–(3.46) is a week output dead-beat controller and not a strict output dead-beat controller for the system (3.42)–(3.44).

3.8 Conclusion

In Chapter 3, we have considered the problem of output dead-beat control with asymptotic disturbance rejection by dynamic compensation. With the aid of dynamic compensator, the arbitrary pole placement can hold when the estimates of the actual state are available as well as via state feedback for the observable and controllable systems. We derived a characterization of all *strict output dead-beat controllers*, which have robustness with respect to disturbance input, by using an eigenvalue–generalized eigenvector assignment method. It has shown that the problem of output dead-beat control with disturbance localization is more restrictive than that of output dead-beat control with asymptotic disturbance rejection.

Chapter 4

ADAPTIVE REGULATION OF LINEAR DISCRETE-TIME SYSTEMS WITH UNKNOWN INPUT

4.1 Introduction

In manufacturing systems and process control systems, the unknown input from the external system with unknown system parameters often causes troubles under operation. We are concerned with the problem of adaptive tracking of uncertain linear systems described by difference equations which contain unknown parameters. The system to be considered here is the control system with unknown input which can be interpreted as the state variable of n_s-th order (it is assumed to be known) state space model with unknown parameters. This state space model of the unknown input is referred to as the unknown input generator. Since we have no information about the structure of the unknown input generator, it is assumed that it has the internal structure so that the n_s dimensional state vector is generated from the time sequence of the scalar white Gaussian noise which is assumed to have zero mean and a unity covariance. Although the actual system structure of the unknown input generator may be different from the derived state space model, the output variables of that state space model can be identified with the output of the actual system of the unknown input generator.

The system to be considered here (see Figure 4.1) is the augmented control system which is composed of the plant to be controlled and the unknown input generator. It is known that, if the plant is observable and some rank condition for the augmented system (Lemma 4.1) is satisfied, there exists a state observer for the augmented state which includes the unknown input [36]. But such a rank condition can not be checked if the system matrix of the unknown input generator A_s is given.

It is well-known that from the controllability of the plant there exists a state feedback gain F which can assign the stable poles as the closed-loop poles of the plant, in case that the state of the plant are directly obtained, and no unknown input is introduced. But if there exists the unknown input in the plant, the output of the plant (which is regarded as a subsystem of the augmented system) cannot be settled to a finite value by using only such a feedback gain F, because the plant state includes the uncertainty. Furthermore, in spite of the observability of the augmented system, the accurate current state of the augmented system can not be obtained from the available system output, either.

For the output regulation problem of the augmented system, first we have to estimate the current state of the augmented system from the available output. The dynamic controller is then constructed based on the estimated state. Here, in order to estimate the state of the unknown input generator, or the partial state of the augmented system, the linear functional observer of the augmented system and the adaptive state observer of the unknown input generator are used. Using parameter identification algorithm and the stable adaptive estimation law, we show that there exists a dynamic controller which can reduce the undesirable effect of the unknown input and stabilize the output of the plant that is the controllable subsystem of the augmented system. The dynamic controller for the augmented system obtained here has two sub-controllers : one is in charge of stabilizing the output of the plant, and the other is in charge of compensating for the unknown input.

The rest of this chapter is organized as follows. In Section 4.2, some preliminary results and definitions are given. The linear functional observer for the augmented system is derived in order to obtain the information about the output of the unknown input generator in Section 4.3. In Section 4.4, we give the structure of adaptive state observer and the adaptive rule of parameter identification. In Section 4.5, we consider that the condition for the existence of the dynamic controller for the augmented system which forces the output to zero asymptotically and keeps it zero for any initial state x_0 independent of the unknown input and furthermore maintain the closed-loop matrix of the controlled system to be stable. Some numerical examples with simulation results are given in Section 4.6.

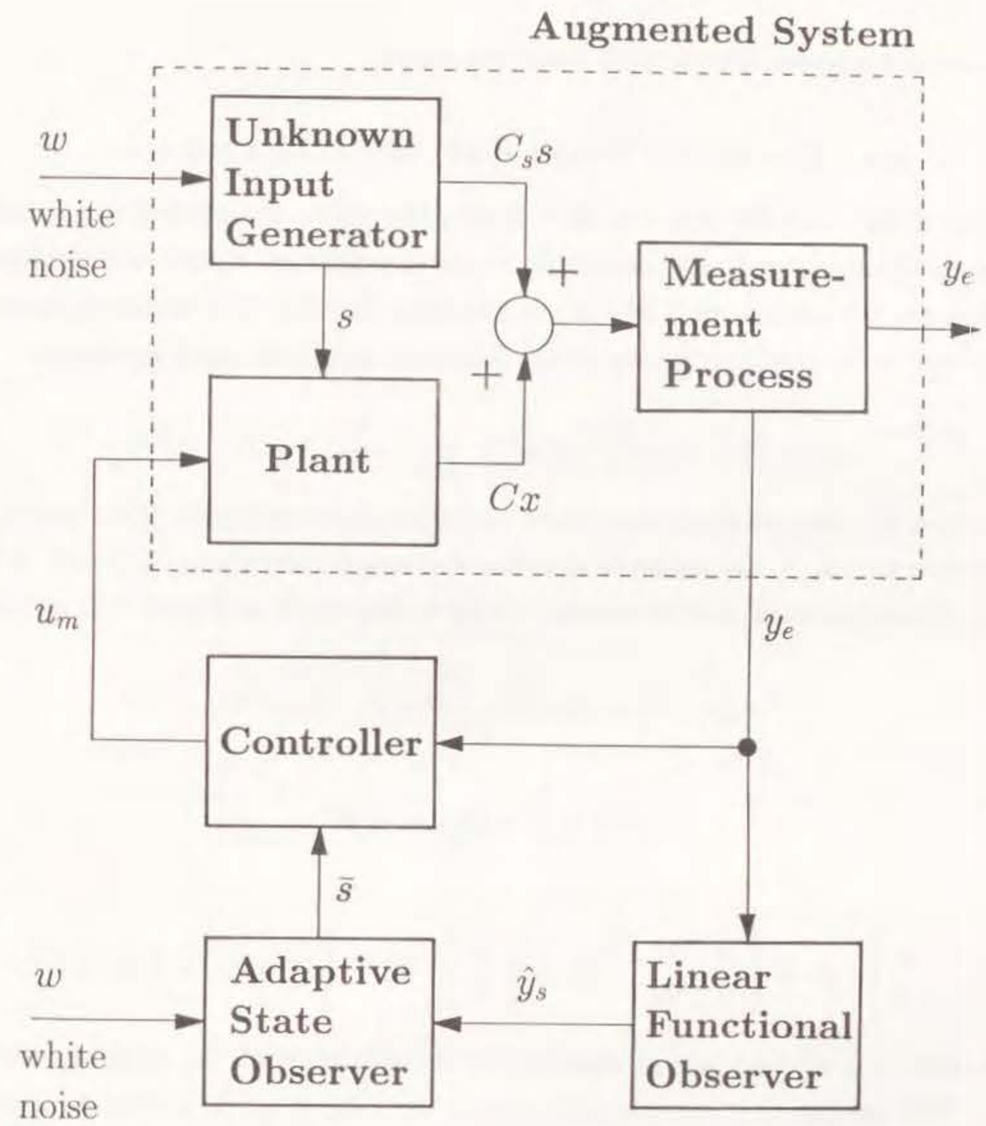

Figure 4.1: Block diagram of a system in which influences of disturbances are reduced by adaptive state observation.

4.2 Problem Statement and Preliminaries

Consider a linear system with unknown input :

$$x(k+1) = Ax(k) + Bu(k) + Es(k),\ x(0) = x_0,\ k = 0, 1, 2, \ldots \tag{4.1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$ and $s \in \mathbf{R}^{n_s}$ denote the state, the control input and the unknown input, respectively, and A, B, E are the constant matrices of appropriate dimensions. We assume that $x(\cdot)$ is not available directly. The unknown input $s(\cdot)$ is assumed to be the state vector of the following unknown input generator :

$$s(k+1) = A_s s(k) + b_s w(k),\ s(0) = s_0,\ k = 0, 1, 2, \ldots \tag{4.2}$$

where $w \in \mathbf{R}^1$ denotes the scalar white Gaussian noise with zero mean and a unity covariance and A_s, b_s are matrices of unknown parameters. From (4.1) and (4.2), we obtain the augmented control system, which is illustrated in Figure 4.2, as follows :

$$x_e(k+1) = A_e x_e(k) + B_e u(k) + b_{se} w(k), \tag{4.3}$$

$$y_e(k) = C_e x_e(k),\ y_e \in \mathbf{R}^p, \tag{4.4}$$

$$x_e = \begin{bmatrix} x \\ s \end{bmatrix}, A_e = \begin{bmatrix} A & E \\ 0 & A_s \end{bmatrix}, B_e = \begin{bmatrix} B \\ 0 \end{bmatrix}, b_{se} = \begin{bmatrix} 0 \\ b_s \end{bmatrix} .C_e = \begin{bmatrix} C & C_s \end{bmatrix}. \tag{4.5}$$

The state $x(\cdot)$ of the plant is regarded as the sub-state of the augmented system (4.3). We assume that we can only obtain the output $y_e(\cdot)$, as the measurement output which includes the output of the unknown input generator.

Now we make the following assumptions :

(A.1) The triplet (C, A, B) is complete.
That is, the pair (A, B) is reachable,
and the pair (C, A) is observable.

(A.2) C has full rank, $C_s \neq 0$, $A_s \neq 0$ and A_s is stable.

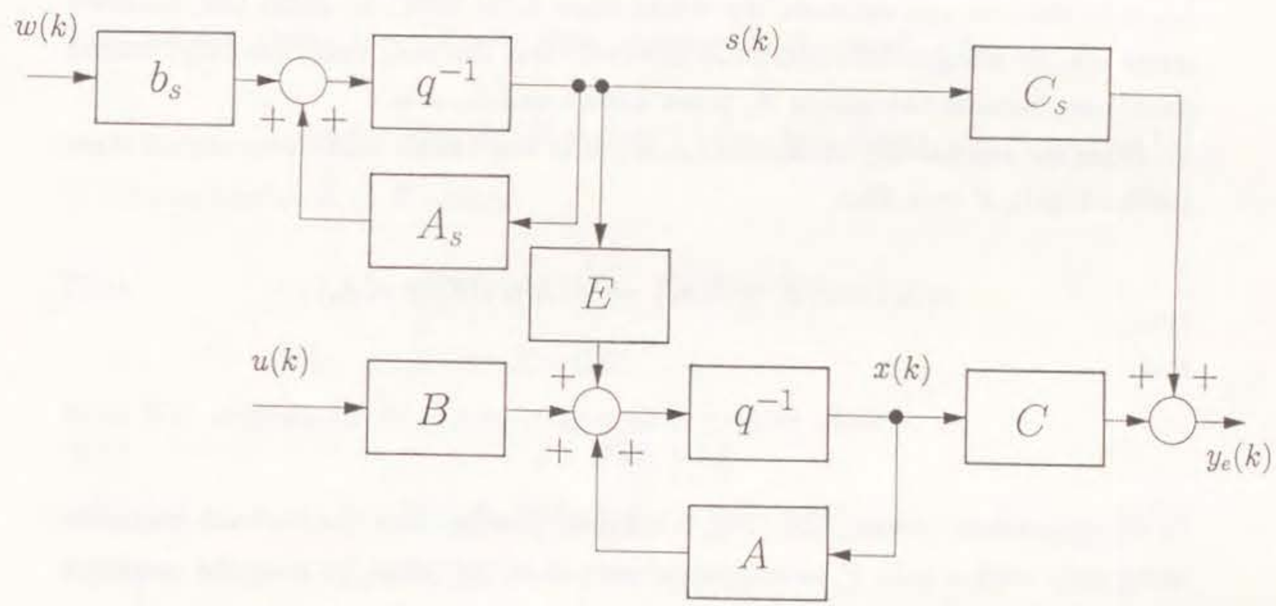

Figure 4.2: Block diagram of the augmented system to be controlled.

It is known that the existence condition of a full order observer for the augmented system (4.3), (4.4) with $b_{se} \equiv 0$ (Lemma 4.1) [36].

Lemma 4.1 *The full order state observer for the augmented system (4.3), (4.4) with $b_{se} \equiv 0$ can be constructed if and only if*

1. *The pair (C, A) is observable.*

2.
$$rank \begin{bmatrix} C & C_s \\ \lambda I - A & -E \\ 0 & \lambda I - A_s \end{bmatrix} = n + n_s , \tag{4.6}$$

where $\lambda \in \sigma(A_s)$.

Assuming **(A.1)**, if (4.6) is satisfied for the augmented system (4.3), (4.4) with $b_{se} \equiv 0$, then we can estimate the whole state $x_e(k)$ which includes the unknown input $s(k)$ by using a full order state observer. But the rank condition (4.6) cannot used here, because the matrix A_s is not known and $b_{se} \neq 0$.

From the reachability of the pair (A, B), it is well-known that there exists a state feedback gain F such that

$$\sigma(A_e) = \sigma(A) \oplus \sigma(A_s) \rightarrow \sigma(A + BF) \oplus \sigma(A_s) , \tag{4.7}$$

and

$$\|A + BF\| < 1 . \tag{4.8}$$

In the augmented system (4.3),(4.4), it is hardly possible that the feedback controller using only such a gain F stabilizes the output of the plant because the unknown input affects the transient and steady state responses of the augmented system. In this chapter, we propose the *modified feedback control* method, in which the above-mentioned feedback gain F and the compensator for the unknown input are used.

In order to use the state feedback gain F such that (4.7), (4.8) is satisfied, the current state $x(k)$ must be reconstructed by using the available input and output data.

(1) The case 1 : C is of column full rank.

For the case that C is of column full rank, which means that the output $y_e(k)$ includes all elements of the plant state $x(k)$, and $p \geq n$, so this case is however

restrictive situation, the current state $x(k)$ of the plant can be reconstructed from (4.4) by using the pseudo-inverse of C

$$x(k) = (C^T C)^{-1} C^T y_e(k) - (C^T C)^{-1} C^T C_s s(k) , \tag{4.9}$$

where the relation $(C^T C)^{-1} C^T C = I_n$ holds. Then, using a feedback gain F which satisfies (4.7) and (4.8), we obtain the feedback control input $u_c(k)$ as follows :

$$\begin{aligned} u_c(k) &= Fx(k) \\ &= F\{(C^T C)^{-1} C^T y_e(k) - (C^T C)^{-1} C^T C_s s(k)\}. \end{aligned} \tag{4.10}$$

(2) The case 2 : C is of row full rank.

For the case that C is a row full rank matrix, we cannot derive the current state $x(k)$ of the plant, because $(C^T C)^{-1} C^T C \neq I_n$. But in this case, as (C, A) is observable, we can construct a full order observer with dead-beat property to estimate the plant state $x(k)$ [36].

Lemma 4.2 *Define the full order state observer for the plant (4.1) :*

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k) + \hat{C}y_e(k) + \hat{E}s(k) . \tag{4.11}$$

If the matrices $\hat{A}, \hat{B}, \hat{C}, \hat{E}$ satisfy

$$\begin{aligned} \hat{A} &= A - \hat{C}C , \quad \hat{A} : nilpotent , \\ \hat{B} &= B , \\ \hat{E} &= E - \hat{C}C_s , \end{aligned} \tag{4.12}$$

then, $\hat{x}(k)$ estimates $x(k)$ exactly after a finite number of steps, i.e.,

$$\hat{x}(\tau + i) = x(\tau + i) , \quad i \geq 0 , \ \tau > 0, \tag{4.13}$$

for some integer τ (Kimura [40], Iwai et al. [36]).

We call the observer (4.11), which satisfies (4.13), *an unbiased observer* or *a dead-beat observer.*

For the case C is of row full rank, we obtain the feedback control input $u_c(k)$ as follows :

$$u_c(k) = F\hat{x}(k) . \tag{4.14}$$

In (4.10)(in the case C has column full rank), $u_c(k)$ includes the unknown input $s(k)$ which is not available. In (4.11)(in the case C has row full rank), $\hat{x}(k)$ can only estimate by using $s(k)$ which is not available either. Furthermore, in the control process, in order to compensate for the unknown input, we must estimate $s(k)$ by other means. We propose the way of estimating $s(k)$ by using the functional observer for the augmented system (4.3), (4.4), and the adaptive state observer for the unknown input generator (4.2).

4.3 Linear Functional Observer for the Augmented System

Now consider a linear functional $y_s(k)$ of the augmented state $x_e(k)$:

$$y_s(k) = l^T x_e(k)\,, \quad y_s \in \mathbf{R}^1\,, \tag{4.15}$$

where

$$l = [\ \underbrace{0 \cdots 0}_{n}\ 1\ 0 \cdots 0\ ]^T \in \mathbf{R}^{n+n_s}.$$

Our problem of this section is to construct a ρ-th order estimator (see Figure 4.3) :

$$z(k+1) = Lz(k) + My_e(k) + Ju(k),\ z \in \mathbf{R}^\rho, \tag{4.16}$$

$$\hat{y}_s(k) = h^T z(k) + g^T y_e(k), \tag{4.17}$$

such that the output $\hat{y}_s(k)$ estimates $y_s(k)(= l^T x_e(k))$ exactly after a finite number of steps, i.e.,

$$\hat{y}_s(\tau + i) = y_s(\tau + i)\,, \quad i \geq 0,\ \tau > 0, \tag{4.18}$$

for some integer τ, i.e., the observer (4.16) and (4.17) is *an unbiased observer.* The following lemma states the existence condition of the unbiased observer for (4.15).

Lemma 4.3 *Assume that the pair (h^T, L) of the linear functional observer (4.16), (4.17) is observable. This linear functional observer is an unbiased observer for (4.15) if and only if there exists a matrix $U_e \in \mathbf{R}^{\rho\times(n+n_s)}$ such that the quintuple (L, M, J, h^T, g^T) satisfies*

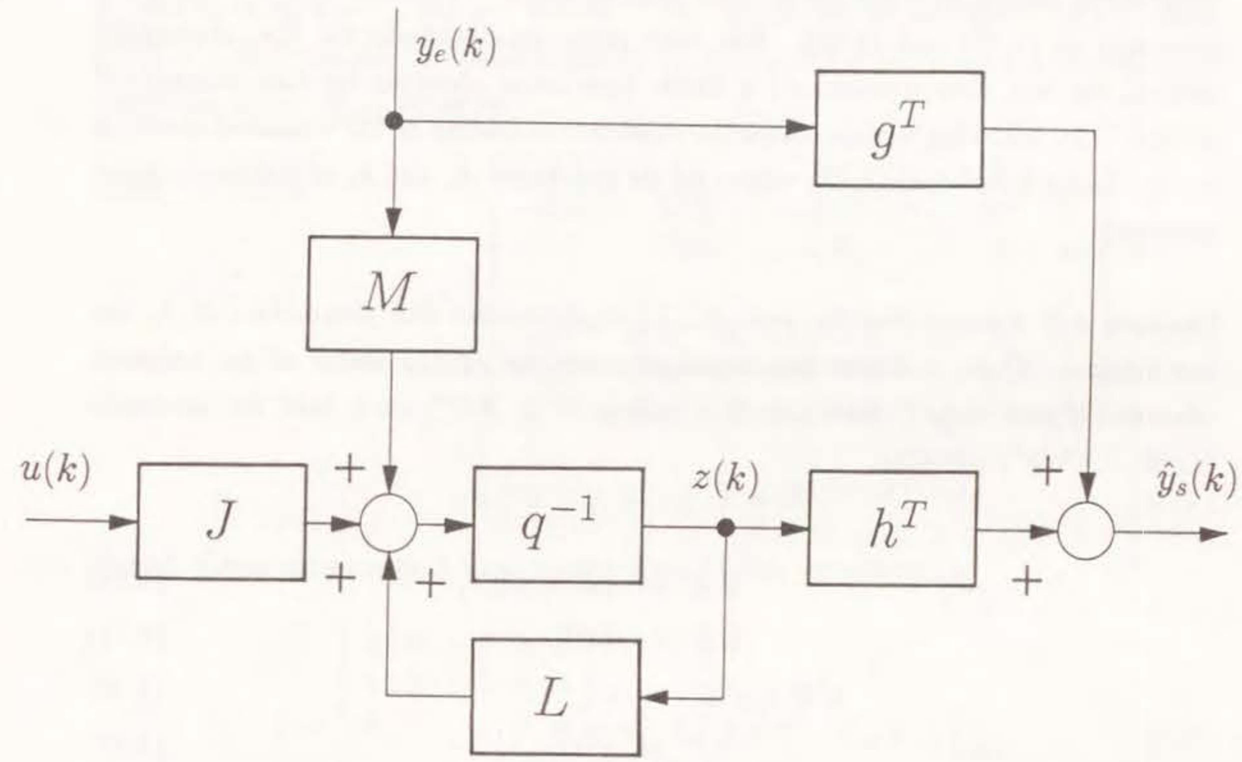

Figure 4.3: Block diagram of the linear functional observer for the augmented system.

$$
\begin{aligned}
U_e A_e &= L U_e + M C_e\,, && (4.19)\\
l^T &= h^T U_e + g^T C_e\,, && (4.20)\\
L^{\rho} &= 0\,, && (4.21)\\
J &= U_e B_e\,, && (4.22)\\
U_e b_{se} &= 0\,. && (4.23)
\end{aligned}
$$

By Lemma 4.3, we cannot check the existence of the unbiased observer for the augmented system because the explicit form of unknown parameters A_s and b_s are necessary in (4.19) and (4.23). But with some specifications for the augmented system, we can always construct a linear functional observer for this augmented system. The following lemma states the existence condition of the unbiased observer for the linear functional (4.15) where we do not know A_s and b_s of unknown input generator.

Lemma 4.4 *Assume that the pair (h^T, L) is observable and parameters A_s, b_s are not known. Then, a linear functional observer for (4.15) exists as an unbiased observer if and only if there exists a matrix $U \in \mathbf{R}^{\rho\times n}$ such that the quintuple (L, M, J, h^T, g^T) satisfies*

$$
\begin{aligned}
UA &= LU + MC\,, && (4.24)\\
UE &= MC_s\,, && (4.25)\\
h^T U + g^T C &= 0\,, && (4.26)\\
\gamma^T &= g^T C_s\,, && (4.27)\\
L^{\rho} &= 0\,, && (4.28)\\
J &= UB\,, && (4.29)
\end{aligned}
$$

where $\gamma = [\,1\,0 \cdots 0\,]^T \in \mathbf{R}^{n_s}$.

We obtain the following.

Theorem 4.1 *Let assume that the augmented system (4.3),(4.4) is observable, and let its observability index be $\rho+1$. Then, for the linear functional (4.15), there exists a ρ dimensional unbiased observer (4.16),(4.17).*

Proof. The solvability of (4.19)-(4.22) in Lemma 4.3 is equivalent to the fact that there exist p dimensional vectors $\xi_1, \xi_2, \cdots, \xi_{\rho+1}$ which satisfy

$$
l^T \varphi(A_e) = \xi_1^T C_e + \xi_2^T C_e A_e + \cdots + \xi_{\rho+1}^T C_e A_e^{\rho}\,, \qquad (4.30)
$$

where $\varphi(\cdot)$ is the characteristic polynomial of L [35]. It is proved that (4.30) is always satisfied if the augmented system is observable. Then, there exists a set of U_e which satisfies (4.19)-(4.22). By selecting U_e of (4.23), we can obtain the quintuple (L, M, J, h^T, g^T) from ξ_i $(i = 1, 2, \cdots, \rho+1)$.

Theorem 4.2 *For the augmented system (4.3),(4.4), let assume that parameters A_s and b_s in A_e and b_{se} are not known. Then, for the linear functional (4.15), there exists a ρ dimensional unbiased observer (4.16),(4.17) if there exist p dimensional vectors $\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_{\rho+1}$ satisfying*

$$
\begin{bmatrix} \tilde{\xi}_1^T & \tilde{\xi}_2^T & \cdots & \tilde{\xi}_{\rho+1}^T \end{bmatrix}
\begin{bmatrix}
C & C_s & 0 & \cdots & 0 & 0\\
CA & CE & C_s & \cdots & 0 & 0\\
CA^2 & CAE & CE & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
CA^{\rho-2} & CA^{\rho-3}E & CA^{\rho-4}E & \cdots & 0 & 0\\
CA^{\rho-1} & CA^{\rho-2}E & CA^{\rho-3}E & \cdots & C_s & 0\\
CA^{\rho} & CA^{\rho-1}E & CA^{\rho-2}E & \cdots & CE & C_s
\end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & \cdots & 0 & \gamma^T \end{bmatrix} \in \mathbf{R}^{1\times\{n+(\rho+1)n_s\}} \qquad (4.31)
$$

Proof. Since the matrix L is nilpotent from (4.28), we may take

$$
L = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0\\
1 & 0 & \cdots & 0 & 0\\
0 & 1 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}_{\rho\times\rho}, \quad h^T = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}_{1\times\rho}. \qquad (4.32)
$$

Let

$$
U := \begin{bmatrix} u_1^T\\ u_2^T\\ \vdots\\ u_\rho^T \end{bmatrix}, \quad M := \begin{bmatrix} m_1^T\\ m_2^T\\ \vdots\\ m_\rho^T \end{bmatrix}, \qquad (4.33)
$$

then, (4.24) and (4.25) become

$$
u_1^T A = m_1^T C\,, \qquad (4.34)
$$

$$u_i^T A = u_{i-1}^T + m_i^T C \ , \ (i = 2, \ldots, \rho) \ , \tag{4.35}$$

$$u_i^T E = m_i^T C_s \ , \ (i = 1, 2, \ldots, \rho) \ . \tag{4.36}$$

Eliminating u_2^T, ..., $u_{\rho-1}^T$ in (4.35), we obtain

$$u_1^T = u_\rho^T A^{\rho-1} - (m_\rho^T C A^{\rho-2} + m_{\rho-1}^T C A^{\rho-3} + \cdots + m_2^T C) \ . \tag{4.37}$$

Substituting (4.37) and the equation $u_\rho^T = -g^T C$ derived from (4.26) into (4.34), it follows that

$$-g^T C A^\rho = m_\rho^T C A^{\rho-1} + m_{\rho-1}^T C A^{\rho-2} + \cdots + m_2^T C A + m_1^T C \ , \tag{4.38}$$

Similarly, substituting (4.37) and the equation $u_\rho^T = -g^T C$ derived from (4.26) into (4.36) at $i = 1$,

$$-g^T C A^{\rho-1} E = m_\rho^T C A^{\rho-2} E + m_{\rho-1}^T C A^{\rho-3} E + \cdots + m_2^T C E + m_1^T C_s \ . \tag{4.39}$$

From (4.38) and (4.39), we obtain

$$-g^T C_e \tilde{A}_e^\rho - m_\rho^T C_e \tilde{A}_e^{\rho-1} - \cdots - m_2^T C_e \tilde{A}_e - m_1^T C_e = 0 \ , \tag{4.40}$$

where

$$\tilde{A}_e := \begin{bmatrix} A & E \\ 0 & 0 \end{bmatrix} .$$

Using the vectors $\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_{\rho+1}$ satisfying the relations of the first column and the second column of (4.31), (4.40) becomes

$$(\tilde{\xi}_{\rho+1}^T - g^T) C_e \tilde{A}_e^\rho + (\tilde{\xi}_\rho^T - m_\rho^T) C_e \tilde{A}_e^{\rho-1} + \cdots + (\tilde{\xi}_2^T - m_2^T) C_e \tilde{A}_e + (\tilde{\xi}_1^T - m_1^T) C_e = 0 \ , \tag{4.41}$$

If we choose

$$g^T = \tilde{\xi}_{\rho+1}^T \ , \tag{4.42}$$
$$m_i^T = \tilde{\xi}_i^T \ , \ (i = 1, 2, \ldots, \rho) \tag{4.43}$$

then (4.42) and (4.43) are the particular solution of (4.41). We obtain the vector u_ρ^T by substituting (4.42) into (4.26). Substituting this vector u_ρ^T and the vectors m_i^T of (4.43) into (4.35), we have $u_{\rho-1}^T, \ldots, u_1^T$ in sequence. Next, we can show that the vectors $u_\rho^T, \ldots, u_1^T$ satisfy (4.36) as follows. Substituting $u_\rho^T, \ldots, u_1^T, m_\rho^T, \ldots, m_1^T$ into (4.36), it follows that

$$\begin{aligned}
\tilde{\xi}_2^T C_s + \tilde{\xi}_3^T C E + \cdots + \tilde{\xi}_{\rho+1}^T C A^{\rho-2} E &= 0 \\
&\vdots \\
\tilde{\xi}_i^T C_s + \tilde{\xi}_{i+1}^T C E + \cdots + \tilde{\xi}_{\rho+1}^T C A^{\rho-i} E &= 0 \\
&\vdots \\
\tilde{\xi}_{\rho-1}^T C_s + \tilde{\xi}_\rho^T C E + \tilde{\xi}_{\rho+1}^T C A E &= 0 \\
\tilde{\xi}_\rho^T C_s + \tilde{\xi}_{\rho+1}^T C E &= 0
\end{aligned}$$

The above equations are satisfied by the relations of the third column, the fourth column, ..., the $(\rho+1)$-th column of (4.31). Furthermore, (4.27) is satisfied by the relation of the $(\rho+2)$-th column of (4.31). Finally, constructing the matrix J from (4.29) in which the vectors $u_\rho^T, \ldots, u_1^T$ are used, we obtain the ρ dimensional unbiased observer (4.16),(4.17).

Remark. Theorem 4.2 holds independent of the observability of the augmented system (4.3),(4.4). If the augmented system (4.3),(4.4) is observable and its observability index is μ_o, then we can take $\rho = \mu_o - 1$ in Theorem 4.2.

By using the estimate $\hat{y}_s(k)$ through the linear functional observer, we will construct the unknown input observer using an adaptive observer.

4.4 Adaptive State Observer for the Unknown Input Generator

The estimate $\hat{y}_s(k)$ of the linear functional (4.15) is regarded as the scalar output of the unknown input generator (4.2). Then, we can rewrite $\hat{y}_s(k)$ as

$$\hat{y}_s(k) = \gamma^T s(k) \ , \tag{4.44}$$

where $\gamma = [1 \, 0 \cdots 0]^T \in \mathbf{R}^{n_s}$. The unknown input generator (4.2) can be described by

$$s(k+1) = \left[\begin{array}{c|c} a_s & \dfrac{d^T}{\Lambda} \end{array} \right] s(k) + b_s w(k) , \tag{4.45}$$

where $d = [1 1 \cdots 1]^T \in \mathbf{R}^{n_s-1}$, $\Lambda = \mathrm{diag}(\lambda_i)$ $(i = 2, 3, \cdots, n_s)$ with $|\lambda_i| < 1$, and $a_s = [a_1, a_2, \cdots, a_{n_s}]$, $b_s = [b_1, b_2, \cdots, b_{n_s}]$ are unknown parameters. Now introducing state variable filters [34], new variables are defined as follows : $(i = 2, 3, \cdots, n_s)$

$$\begin{aligned} v_i(k+1) &= \lambda_i v_i(k) + \hat{y}_s(k), \\ t_i(k+1) &= \lambda_i t_i(k) + w(k), \end{aligned} \tag{4.46}$$

where $|\lambda_i| < 1$ and $\lambda_i \neq \lambda_j$ if $i \neq j$.

Using (4.46), from (4.45), it follows that

$$\begin{aligned} s_i(k) &= a_i v_i(k) + b_i t_i(k) + (\lambda_i)^k s_{0i}, \\ &\quad (s_{0i} = s_i(0),\ i = 2, 3, \cdots, n_s) \end{aligned}$$

$$\begin{aligned} \hat{y}_s(k+1) &= \lambda_1 \hat{y}_s(k) + (a_1 - \lambda_1)\hat{y}_s(k) + \sum_{i=2}^{n_s} a_i v_i(k) \\ &\quad + b_1 w(k) + \sum_{i=2}^{n_s} b_i t_i(k) + \sum_{i=2}^{n_s} (\lambda_i)^k s_{0i}, \end{aligned}$$

where $|\lambda_1| < 1$ and $s(k) = [\hat{y}_s(k), s_2(k), \cdots, s_{n_s}(k)]$.

Now the adaptive state observer is defined as

$$\begin{aligned} \bar{s}_i(k) &= \bar{a}_i(k) v_i(k) + \bar{b}_i t_i(k) + (\lambda_i)^k \bar{s}_{0i} , \\ &\quad (\bar{s}_{0i} = \bar{s}_i(0),\ i = 2, 3, \cdots, n_s) \end{aligned} \tag{4.47}$$

$$\begin{aligned} \bar{y}_s(k+1) &= \lambda_1 \bar{y}_s(k) + \{\bar{a}_1(k+1) - \lambda_1\}\hat{y}_s(k) + \sum_{i=2}^{n_s} \bar{a}_i(k+1) v_i(k) \\ &\quad + \bar{b}_1(k+1) w(k) + \sum_{i=2}^{n_s} \bar{b}_i(k+1) t_i(k) + \sum_{i=2}^{n_s} (\lambda_i)^k \bar{s}_{0i}. \end{aligned} \tag{4.48}$$

Define $e_1(k) := \bar{y}_s(k) - \hat{y}_s(k)$. Then the error equation is

$$e_1(k+1) = \lambda_1 e_1(k) + \theta^T(k+1) q(k) + f_0(k) , \tag{4.49}$$

where

$$\theta^T(k) := [\,\zeta^T(k), \eta^T(k)\,] ,\ q^T(k) := [\,v^T(k), t^T(k)\,] ,$$

and

$$\zeta(k) := \begin{bmatrix} \bar{a}_1(k) - a_1 \\ \bar{a}_2(k) - a_2 \\ \vdots \\ \bar{a}_{n_s}(k) - a_{n_s} \end{bmatrix} ,\ \eta(k) := \begin{bmatrix} \bar{b}_1(k) - b_1 \\ \bar{b}_2(k) - b_2 \\ \vdots \\ \bar{b}_{n_s}(k) - b_{n_s} \end{bmatrix} ,$$

$$v(k) := \begin{bmatrix} \hat{y}_s(k) \\ \bar{v}(k) \end{bmatrix} ,\ \bar{v}(k) := \begin{bmatrix} v_2(k) \\ \vdots \\ v_{n_s}(k) \end{bmatrix} ,\ t(k) := \begin{bmatrix} w(k) \\ \bar{t}(k) \end{bmatrix} ,\ \bar{t}(k) := \begin{bmatrix} t_2(k) \\ \vdots \\ t_{n_s}(k) \end{bmatrix} ,$$

$$f_0(k) := \sum_{i=2}^{n_s} (\lambda_i)^k (\bar{s}_{0i} - s_{0i}) .$$

Then, the adaptive rule for parameter identification is given by

$$\theta(k+1) = \theta(k) - \frac{\alpha\delta q(k-1)\theta^T(k) q(k-1)}{\lambda_{max} q^T(k-1) q(k-1)} , \tag{4.50}$$

where $0 < \alpha < 2$, δ is a real symmetric positive matrix and λ_{max} is the maximal eigenvalue of δ. Furthermore, the parameter adaptive law is

$$\begin{bmatrix} \bar{a}^T(k+1) \\ \bar{b}^T(k+1) \end{bmatrix} = \begin{bmatrix} \bar{a}^T(k) \\ \bar{b}^T(k) \end{bmatrix} - \frac{\alpha\delta[e_1(k) - \lambda_1 e_1(k-1)]}{\lambda_{max} q^T(k-1) q(k-1)} q(k-1) . \tag{4.51}$$

The above-mentioned design method of the adaptive observer is similar to the design method of Lüders-Narendra type adaptive observer [51].

Since $\bar{s}_i(k)$ is given from (4.47) by using identified parameters and $\bar{y}_s(k)$ is given from (4.48), we can obtain the estimate $\bar{s}(k) = [\,\bar{y}_s(k),\ \bar{s}_2(k),\ \cdots,\ \bar{s}_{n_s}(k)\,]$ of the unknown input $s(k)$.

4.5 Dynamic Output Control

In this section, we consider the control problem for the augmented system (4.3) so that the controlled output is driven to zero asymptotically and is kept zero independent of the unknown input, and furthermore the closed-loop matrix of the system (4.1) is maintained to be stable. We can construct a dynamic controller for the system (4.3) by compensating for the unknown input.

Theorem 4.3 *Assume that the augmented system (4.3), (4.4) satisfies (i)* **(A.1)**, *(ii)* **(A.2)**. *Now consider the output control problem in which the output to be stabilized to zero is given by*

$$\begin{aligned} y_o(k) &= D\,x(k), & (4.52) \\ &= [D \quad 0]x_e(k), & (4.53) \end{aligned}$$

and the following input is used as a control input :

1. *For the case that C is a column full rank matrix,*

$$\begin{aligned} u_m(k) &= \bar{u}_c(k) + T\bar{s}(k) \\ &= F\{(C^TC)^{-1}C^Ty_e(k) - (C^TC)^{-1}C^TC_s\bar{s}(k)\} + T\bar{s}(k)\,. \end{aligned} \qquad (4.54)$$

2. *For the case that C is a row full rank matrix,*

$$\begin{aligned} u_m(k) &= u_c(k) + T\bar{s}(k) \\ &= F\hat{x}(k) + T\bar{s}(k)\,, \end{aligned} \qquad (4.55)$$

where $\hat{x}(k)$ is the estimate of the system state $x(k)$ derived from the full order state observer (4.11).

In (4.54) and (4.55), $\bar{s}(k)$ is the estimate of the unknown input $s(k)$ which is the state of the unknown input generator (4.2), and $\bar{s}(k)$ is obtained by using a linear functional unbiased observer (4.16),(4.17) which is given by Theorem 4.2 and an adaptive state observer (4.47),(4.48). Then, this modified control input $u_m(k)$ can drive the output (4.52) to zero asymptotically, and maintain the internal stability of the system to be controlled (4.1) if and only if there exist the matrices $F \in \mathbf{R}^{r\times n}$ and $T \in \mathbf{R}^{r\times n_s}$ such that the following two conditions are satisfied.

1. *The feedback control gain F, which satisfies (4.7) and (4.8), meets the following :*

$$(A + BF)\mathcal{V}^* \subset \mathcal{V}^*\,, \qquad (4.56)$$

where

$$\mathcal{V}^* := sup\,\underline{\mathcal{I}}(A\,,\,B\,;\,Ker\,D)\,,$$

that is, $\mathcal{V}^*$ is the maximal $(A\,,\,B)$-invariant subspace included in $Ker\,D$ (see (1.19) [85]).

2. *The compensator gain T satisfies*

$$Im\,(E + BT) \subset \mathcal{V}^*\,. \qquad (4.57)$$

Proof.

(1) The proof for the case that C is of column full rank is as follows.

Substituting $u_m(k)$ of (4.54) for $u(k)$ in (4.3), we obtain

$$\begin{aligned} x_e(k+1) &= A_ex_e(k) + B_eu_m(k) + b_{se}w(k), \\ &= A_ex_e(k) + B_e\{F(C^TC)^{-1}C^TC_ex_e(k) - F(C^TC)^{-1}C^TC_s\bar{s}(k)\} \\ &\quad + B_eT\bar{s}(k) + b_{se}w(k)\,. \end{aligned} \qquad (4.58)$$

From (4.58), the state $x(k)$ of the system (4.1) is described by

$$x(k+1) = (A+BF)x(k) + Es(k) + BT\bar{s}(k) + BF(C^TC)^{-1}C^TC_s\{s(k) - \bar{s}(k)\}\,, \qquad (4.59)$$

where the relation $(C^TC)^{-1}C^TC = I_n$ is used. In (4.59), it is assumed that $\bar{s}$ is the unbiased estimate of a disturbance s, that is, for some integer $N\,(> 0)$,

$$\bar{s}(k+N) = s(k+N)\,,\ (k > 0\,,\ N > 0) \qquad (4.60)$$

(Necessity) In order to maintain the internal stability of the system (4.59), the closed-loop system map must be stable, that is, $\|A+BF\| < 1$. From the reachability of the pair $(A\,,\,B)$ (**(A.1)**), we can always choose such a feedback gain F that (4.8) holds independent of A_s ((4.7)). Substituting (4.59) into (4.52), it follows that

$$\begin{aligned} y_o(k) &= Dx(k) \\ &= D[(A+BF)^kx_0 \\ &\quad + \textstyle\sum_{i=0}^{k-1}(A+BF)^{k-1-i}Es(i) \\ &\quad + \textstyle\sum_{i=0}^{k-1}(A+BF)^{k-1-i}BT\bar{s}(i) \\ &\quad + \textstyle\sum_{i=0}^{k-1}(A+BF)^{k-1-i}BF(C^TC)^{-1}C^TC_s\{s(k) - \bar{s}(i)\}]\,. \end{aligned} \qquad (4.61)$$

Since the control input $u_m(k)$ can drive the output $y_o(k)$ of (4.52) to zero asymptotically, $y_o(k)$ must converge to zero as $k \to \infty$. The first term of the right-hand side in (4.61) converges to zero for any initial state x_0, that is,

$$(A+BF)^k x_0 \longrightarrow 0 \qquad \text{as } k \to \infty\,,$$

by using F such that $\|A+BF\| < 1$. From (4.60), it follows that

$$|\bar{s}(k+N) - s(k+N)| = 0\,,\ (k > 0)$$

for some integer $N > 0$. Then, $y_o(N+k)$ becomes

$$\begin{array}{rcl} y_o(N+k) & = & D[(A+BF)^{N+k}x_0 \\ & & +\sum_{i=0}^{N+k-1}(A+BF)^{N+k-1-i}\{Es(i)+BT\bar{s}(i)\} \\ & & +\sum_{i=0}^{N}(A+BF)^{N+k-i}BF(C^TC)^{-1}C^TC_s\{s(i)-\bar{s}(i)\}] \end{array} \tag{4.62}$$

The third term of the right-hand side in (4.62) is the finite summation with respect to the term

$$\delta(i) := BF(C^TC)^{-1}C^TC_s\{s(i)-\bar{s}(i)\}\,,\ i = 0,1,\ldots,N,$$

therefore

$$\sum_{i=0}^{N}(A+BF)^{N+k-i}\delta(i) \longrightarrow 0 \qquad \text{as } k \to \infty,$$

by using the stable closed-loop matrix $A+BF$.

As the second term of the right-hand side in (4.62) is described by

$$\begin{array}{rl} & \sum_{i=0}^{N+k-1}(A+BF)^{N+k-1-i}\{Es(i)+BT\bar{s}(i)\} \\ = & \sum_{i=N+1}^{N+k-1}(A+BF)^{N+k-1-i}(E+BT)s(i) \\ & +\sum_{i=0}^{N}(A+BF)^{N+k-1-i}\{Es(i)+BT\bar{s}(i)\}\,, \end{array} \tag{4.63}$$

it follows that

$$\sum_{i=0}^{N}(A+BF)^{N+k-1-i}\{Es(i)+BT\bar{s}(i)\} \longrightarrow 0\,, \tag{4.64}$$

as $k \to \infty$. Finally, in order to drive the controlled output y_o to zero asymptotically, it is necessary to satisfy the following condition

$$\sum_{i=N+1}^{N+k-1}(A+BF)^{N+k-1-i}(E+BT)s(i) \in \operatorname{Ker} D\,, \tag{4.65}$$

as $k \to \infty$. The condition (4.65) is the one of the disturbance decoupling with respect to the term $(E+BT)s(i)$. The disturbance decoupling problem for the condition (4.65) is solvable if and only if there exists a matrix T satisfying (4.57), in which $\mathcal{V}^*$ must be chosen such that (4.56) is satisfied because the closed-loop system map must be stable.

(Sufficiency) Suppose that F and T of (4.54) are chosen such that the conditions 1 and 2 are satisfied. Substituting $u_m(k)$ of (4.54) for $u(k)$ in (4.3), we obtain the controlled output y_o as (4.62). From the stability of the closed-loop matrix $A+BF$ and (4.60), the first and the third term of the right-hand side in (4.62) are driven to zero as $k \to \infty$. The second term of the right-hand side in (4.62) is rewritten as (4.63), in which the second term of the right-hand side is driven to zero as $k \to \infty$. Furthermore, the first term of the right-hand side in (4.63) is decoupled relative to the output y_o from (4.57). Thus, the control input u_m of (4.54) drives the output to zero asymptotically and maintains the internal stability of the system (4.1).

(2) The proof for the case that C is of row full rank is as follows.

Substituting $u_m(k)$ of (4.55) for $u(k)$ in (4.3), we obtain

$$\begin{array}{rcl} x_e(k+1) & = & A_e x_e(k) + B_e u_m(k) + b_{se}w(k), \\ & = & A_e x_e(k) + B_e F\hat{x}(k) + B_e T\bar{s}(k) + b_{se}w(k)\,. \end{array} \tag{4.66}$$

From (4.66), the state $x(k)$ of the system (4.1) is described by

$$x(k+1) = Ax(k) + BF\hat{x}(k) + Es(k) + BT\bar{s}(k)\,. \tag{4.67}$$

Using $e_x(k) := \hat{x}(k) - x(k)$, (4.67) is rewritten as

$$x(k+1) = (A+BF)x(k) + BFe_x(k) + Es(k) + BT\bar{s}(k)\,. \tag{4.68}$$

In (4.68), it is assumed that $\hat{x}(k)$, $\bar{s}(k)$ are the unbiased estimates of the system state $x(k)$ and a external disturbance $s(k)$ respectively, that is, for some integers N_1, N_2 (> 0),

$$\begin{array}{rcl} \hat{x}(k+N_1) & = & x(k+N_1),\ \ e_x(k+N_1) = 0\,,\ (k>0\,,\ N_1>0)\,, \\ \bar{s}(k+N_2) & = & s(k+N_2),\ \ (k>0\,,\ N_2>0)\,. \end{array} \tag{4.69}$$

Taking $N > 0$ such that $N := \max(N_1\,,\ N_2)$, then (4.68) becomes

$$x(k+1) = (A+BF)x(k) + (E+BT)s(k)\,,\ k \geq N\,. \tag{4.70}$$

The open-loop system poles of the augmented system (4.3), (4.4) with the full order state observer (4.11) is described by $\sigma(A_e) \oplus \sigma(\hat{A})$. Then, in order to maintain the internal stability of the system (4.70), the closed-loop system map must be stable, that is, $\|A+BF\| < 1$, because A_s is stable ((**A.2**)) and $\hat{A}$ is nilpotent (Lemma 4.2),

$$\sigma(A_e) \oplus \sigma(\hat{A}) \rightarrow \sigma(A+BF) \oplus \sigma(A_s) \oplus \sigma(\hat{A}) .$$

Thus, the necessity and sufficiency is proved in the same way as the case that C is of column full rank.

4.6 Numerical Examples

Example 4.1.

Consider the plant with unknown input :

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} s(k) , \tag{4.71}$$

where the unknown input $s(k)$ is assumed to be the state variable of the unknown input generator of dimension 3 which is assumed to be known. Although A is unstable ($\sigma(A) = \{\pm i\}$, where i is the purely imaginary number), we can choose a feedback control gain F such that $\sigma(A+BF) = \{\pm 0.5\}$ which is a set of stable closed-loop poles because the plant (4.71) is reachable :

$$F = \begin{bmatrix} 0 & 0 \\ 1.25 & 0 \end{bmatrix} . \tag{4.72}$$

The augmented system is given by

$$x_e(k+1) = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0_{3\times 2}} & \multicolumn{3}{c}{A_s} \end{array}\right] x_e(k) + \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline \multicolumn{2}{c}{0_{3\times 2}} \end{array}\right] u(k) + \left[\begin{array}{c} 0_2 \\ \hline b_s \end{array}\right] w(k) , \tag{4.73}$$

$$y_e(k) = \left[\begin{array}{cc|ccc} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] x_e(k) . \tag{4.74}$$

In this example, C has column full rank.

Let

$$y_s(k) = [0\,0\,1\,0\,0]\, x_e(k) . \tag{4.75}$$

Although we don't know the augmented system (4.73),(4.74) is observable or not, it may be considered that the order of the linear functional observer, ρ, is less than or equal to 4, from the observability matrix, and the plant is 2 dimensional observable system, we can assume that ρ is 2 or 3. Furthermore, as we have assumed that the order of the unknown input generator is 3, we can assume that $\rho = 2$ without any restriction. According to Theorem 4.2, such three dimensional vectors $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3$ which satisfy (4.31) are given by

$$\tilde{\xi}_1 = \begin{bmatrix} 0 \\ -0.25 \\ -0.25 \end{bmatrix} , \tilde{\xi}_2 = \begin{bmatrix} 1.5 \\ 0.25 \\ -1 \end{bmatrix} , \tilde{\xi}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} . \tag{4.76}$$

Therefore, using (4.76) and Lemma 4.4, we obtain the linear functional observer of 2 dimension as follows:

$$U = \begin{bmatrix} -0.5 & -0.25 \\ -1 & -1 \end{bmatrix} \text{ (from Lemma 4.4)} , \tag{4.77}$$

$$z(k+1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z(k) + \begin{bmatrix} 0 & -0.25 & -0.25 \\ 1.5 & 0.25 & -1 \end{bmatrix} y_e(k) + \begin{bmatrix} -0.5 & -0.25 \\ -1 & -1 \end{bmatrix} u(k) , \tag{4.78}$$

$$\hat{y}_s(k) = [0 \;\; 1] z(k) + [1 \;\; 1 \;\; 0] y_e(k) . \tag{4.79}$$

From (4.45), the unknown input generator can be written as

$$s(k+1) = \left[\begin{array}{c|cc} & 1 & 1 \\ a_s & -0.5 & 0 \\ & 0 & 0 \end{array}\right] s(k) + b_s w(k) , \tag{4.80}$$

where $a_s = [a_1\; a_2\; a_3]^T$ and $b_s = [b_1\; b_2\; b_3]^T$ are unknown parameters. Introducing state variable filters according to (4.46), new variables are defined as

$$\begin{aligned} v_2(k+1) &= -0.5 v_2(k) + \hat{y}_s(k), \\ t_2(k+1) &= -0.5 t_2(k) + w(k). \end{aligned} \tag{4.81}$$

$$v_3(k+1) = \hat{y}_s(k) , \; t_3(k+1) = w(k) . \tag{4.82}$$

where we set $\lambda_2 = -0.5$ and $\lambda_3 = 0$. Using (4.81) and (4.82), it follows from (4.80) that

$$\begin{aligned} s_2(k) &= a_2 v_2(k) + b_2 t_2(k) + (-0.5)^k s_{02}, \\ s_3(k) &= a_3 v_3(k) + b_3 t_3(k), \\ \hat{y}_s(k+1) &= 0.25\hat{y}_s(k) + (a_1 - 0.25)\hat{y}_s(k) + \textstyle\sum_{i=2}^{3} a_i v_i(k) \\ &\quad + b_1 w(k) + \textstyle\sum_{i=2}^{3} b_i t_i(k) + (-0.5)^k s_{02}, \end{aligned} \tag{4.83}$$

where we set $\lambda_1 = 0.25$. Now the adaptive state observer for (4.83) is defined as follows :

$$\bar{s}(k) = [\,\bar{y}_s(k)\ \ \bar{s}_2(k)\ \ \bar{s}_3(k)\,]^T, \tag{4.84}$$

where

$$\begin{aligned} \bar{s}_2(k) &= \bar{a}_2 v_2(k) + \bar{b}_2 t_2(k) + (-0.5)^k \bar{s}_{02}, \\ \bar{s}_3(k) &= \bar{a}_3 v_3(k) + \bar{b}_3 t_3(k), \\ \bar{y}_s(k+1) &= 0.25\bar{y}_s(k) + \{\bar{a}_1(k+1) - 0.25\}\hat{y}_s(k) \\ &\quad + \textstyle\sum_{i=2}^{3} \bar{a}_i(k+1) v_i(k) + b_1(k+1) w(k) \\ &\quad + \textstyle\sum_{i=2}^{3} \bar{b}_i(k+1) t_i(k) + (-0.5)^k \bar{s}_{02}. \end{aligned} \tag{4.85}$$

Define $e_1(k) := \bar{y}_s(k) - \hat{y}_s(k)$. Then the error equation is

$$\begin{aligned} e_1(k+1) &= 0.25 e_1(k) + \{\bar{a}_1(k+1) - a_1\}\hat{y}_s(k) \\ &\quad + \sum_{i=2}^{3} \{\bar{a}_i(k+1) - a_i\} v_i(k) + \{\bar{b}_1(k+1) - b_1\} w(k) \\ &\quad + \sum_{i=2}^{3} \{\bar{b}_i(k+1) - b_i\} t_i(k) + (-0.5)^k (\bar{s}_{02} - s_{02}). \end{aligned} \tag{4.86}$$

In (4.86), unknown parameters $\{\bar{a}_i(k) - a_i\}, \{\bar{b}_i(k) - b_i\}$ $(i = 1, 2, 3)$ are identified by the adaptive rule (4.50) and $\bar{a}_s(k), \bar{b}_s(k)$ are identified by the adaptive law (4.51).

Taking, from (4.52),

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then, the feedback gain F (4.72) satisfies the condition 1 of Theorem 4.3. Using the estimate $\bar{s}(k)$ (in (4.84)) and choosing T which satisfies the condition 2 of Theorem 4.3 as

$$T = -E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \tag{4.87}$$

finally, we obtain the modified control input $u_m(k)$ from (4.54) as

$$u_m(k) = \begin{bmatrix} 0 & 0 & 0 \\ 1.25 & 0 & 0 \end{bmatrix} y_e(k) + \begin{bmatrix} -1 & 0 & 0 \\ 1.25 & -1 & 0 \end{bmatrix} \bar{s}(k). \tag{4.88}$$

This control input $u_m(k)$ is the state dead-beat controller for the plant (4.71). Figure 4.4 and 4.5 show the behavior of the output $y_o(k)$ of the plant (4.71) based on the feedback gain F (given by (4.72)) under unknown input. Figure 4.6 and 4.7 show the behavior of the output $y_o(k)$ of the plant (4.71) based on the proposed controller $u_m(k)$ (given by (4.88)) under unknown input, where the state $x(k)$ is forced to zero from the initial state $x_0 = [1\ \ 1]^T$. Figure 4.8 shows the observation process of the linear functional y_s, in which the estimate $\hat{y}_s$ presents the unbiased estimate of y_s. The estimation process of the disturbance $s(k)$ is shown in Figure 4.9, 4.10 and 4.11. Figure 4.12 and Figure 4.13 show the estimation processes of the unknown parameters a_s and b_s in (4.80). In this example, the true values of a_s and b_s are $a_s = [-1\ \ -0.5\ \ 0.1]^T$ and $b_s = [0.5\ \ -0.25\ \ 0]^T$. Finally, Figure 4.14 shows the transient response of the error equation (4.86), which shows the adaptive estimation error tends to zero as time-steps.

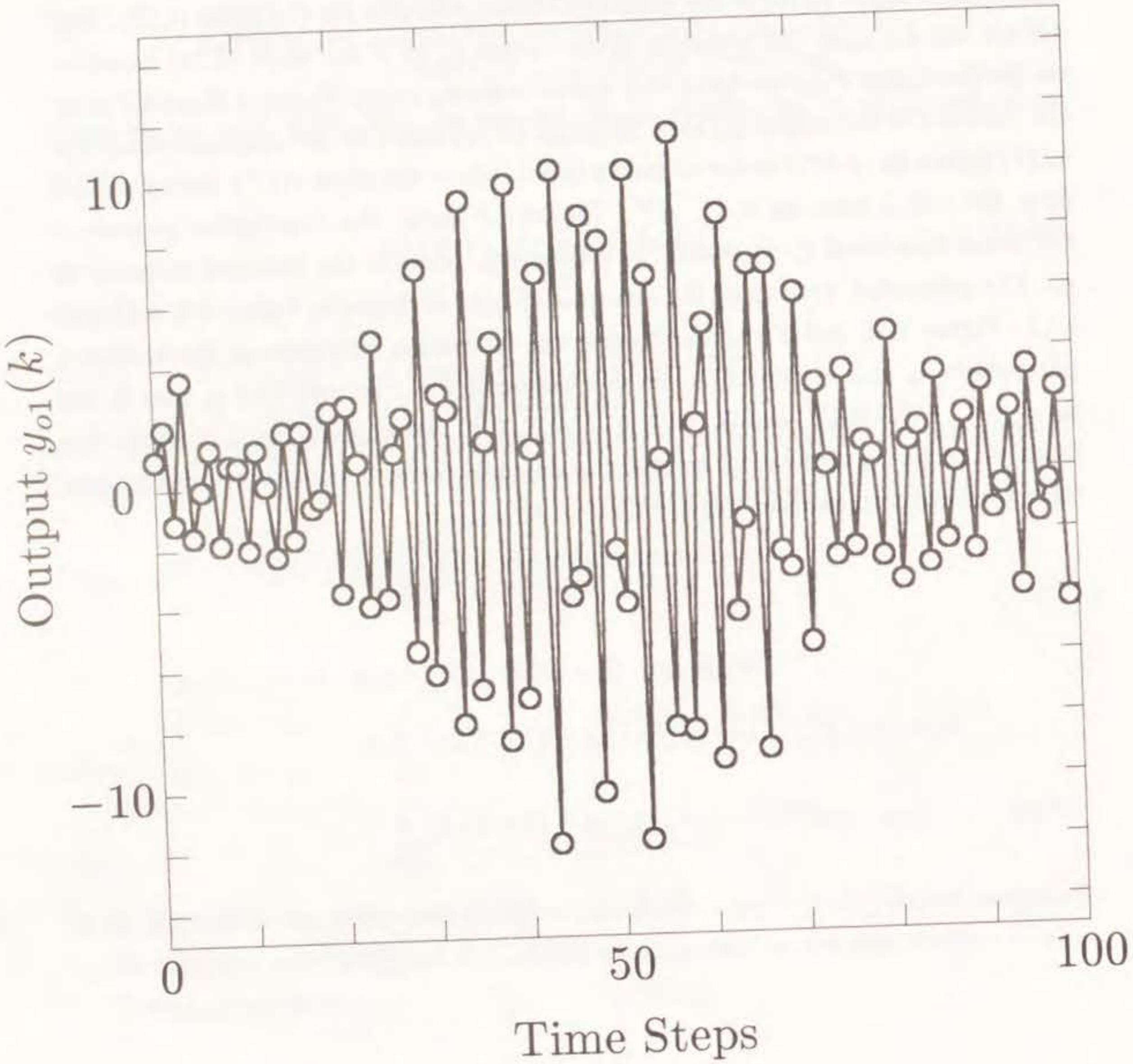

Figure 4.4: Behavior of the first element $y_{o1}(k)$ of the output $y_o(k)$ based on the feedback gain F(Example 4.1).

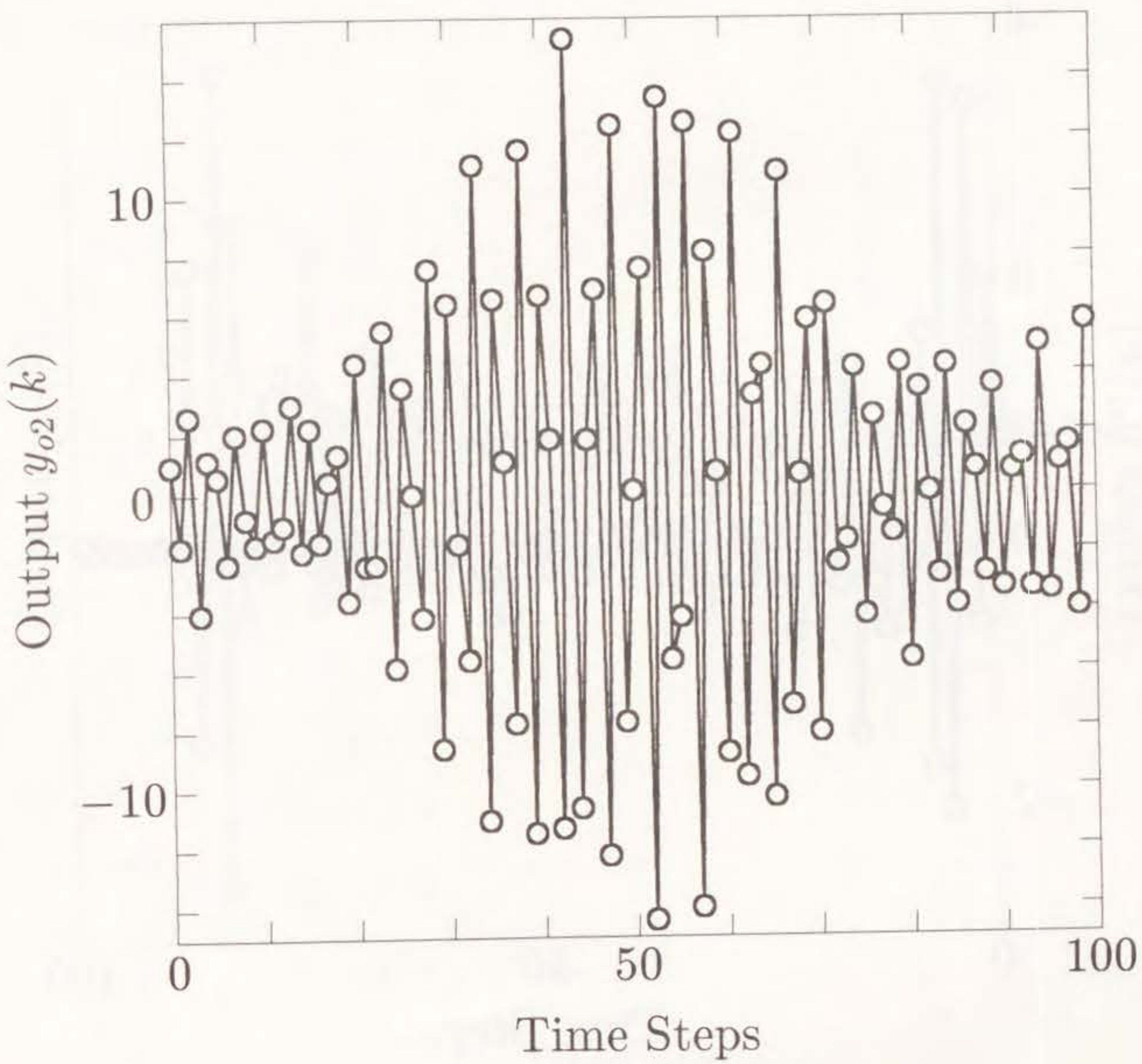

Figure 4.5: Behavior of the second element $y_{o2}(k)$ of the output $y_o(k)$ based on the feedback gain F(Example 4.1).

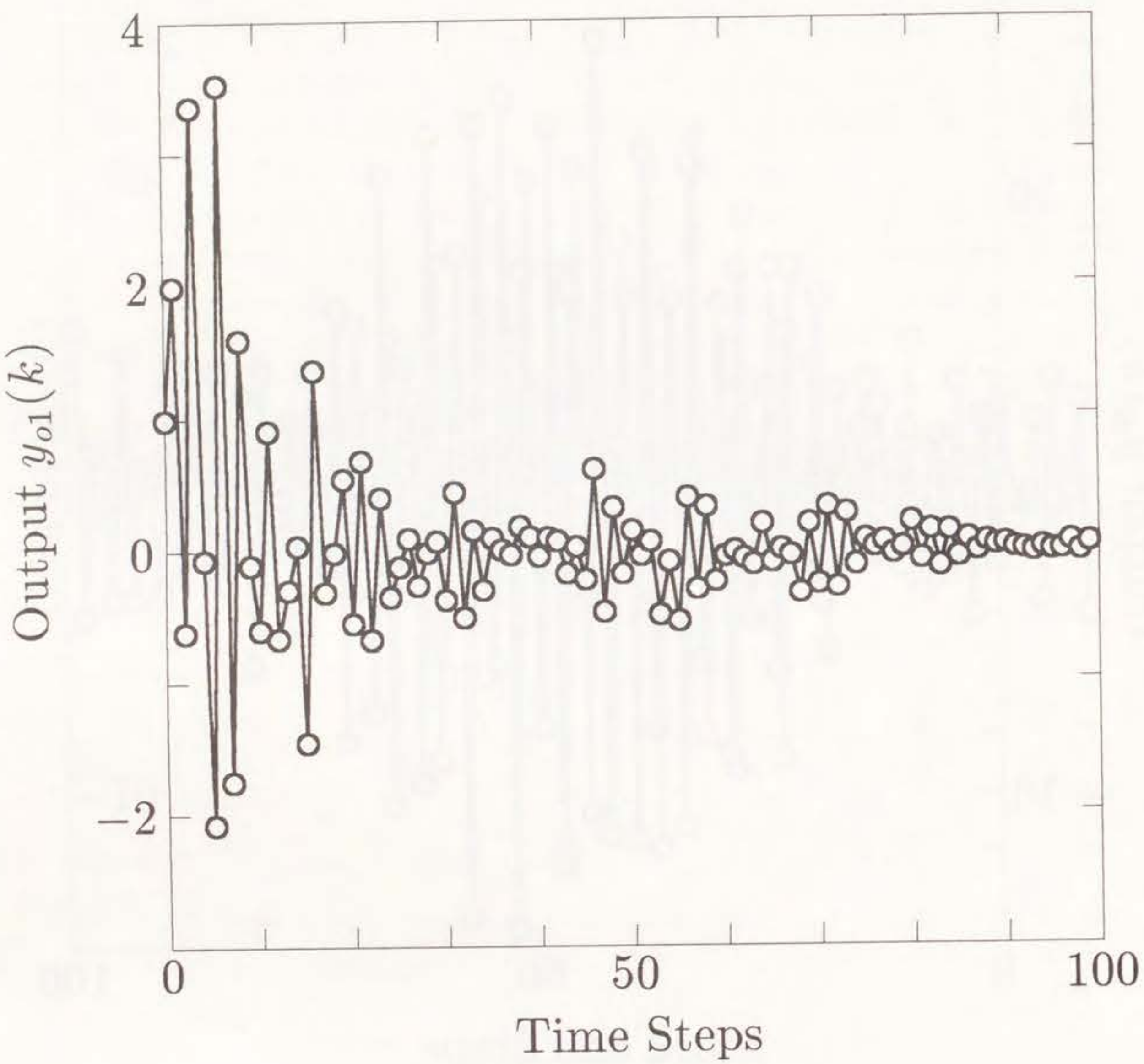

Figure 4.6: Behavior of the first element $y_{o1}(k)$ of the output $y_o(k)$ based on the proposed controller $u_m(k)$(Example 4.1).

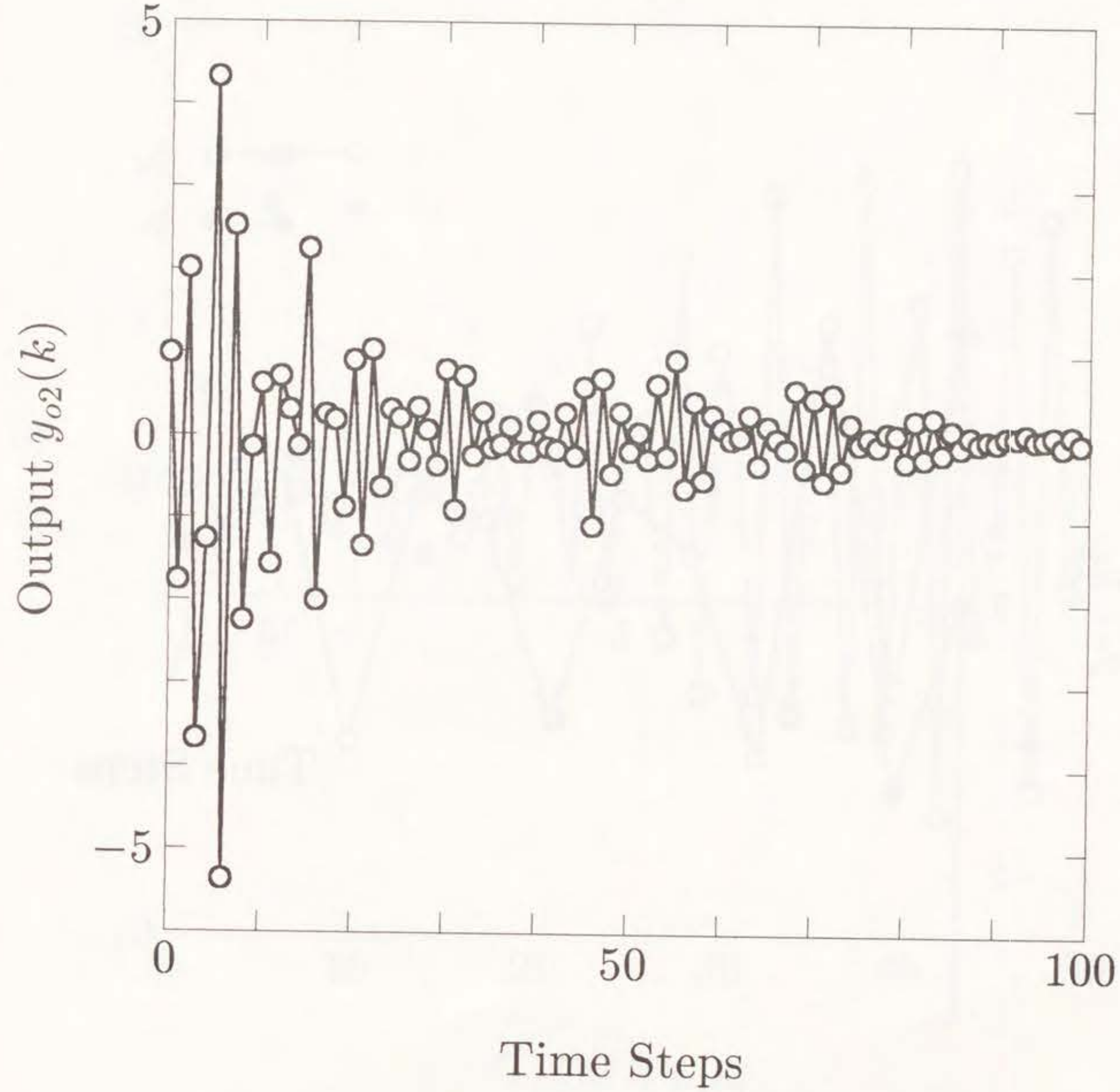

Figure 4.7: Behavior of the second element $y_{o2}(k)$ of the output $y_o(k)$ based on the proposed controller $u_m(k)$(Example 4.1).

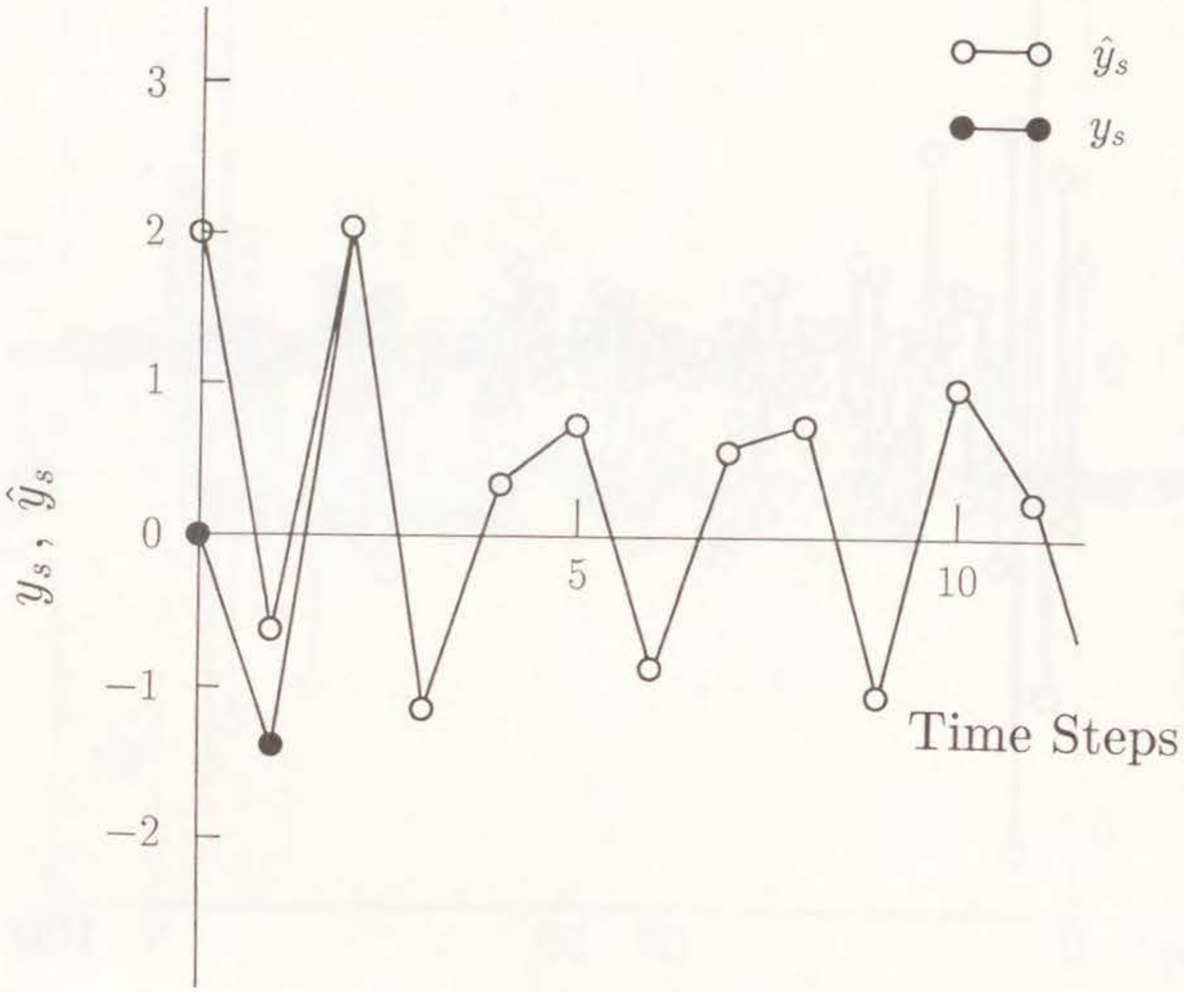

Figure 4.8: Observation process of the linear functional y_s(Example 4.1).

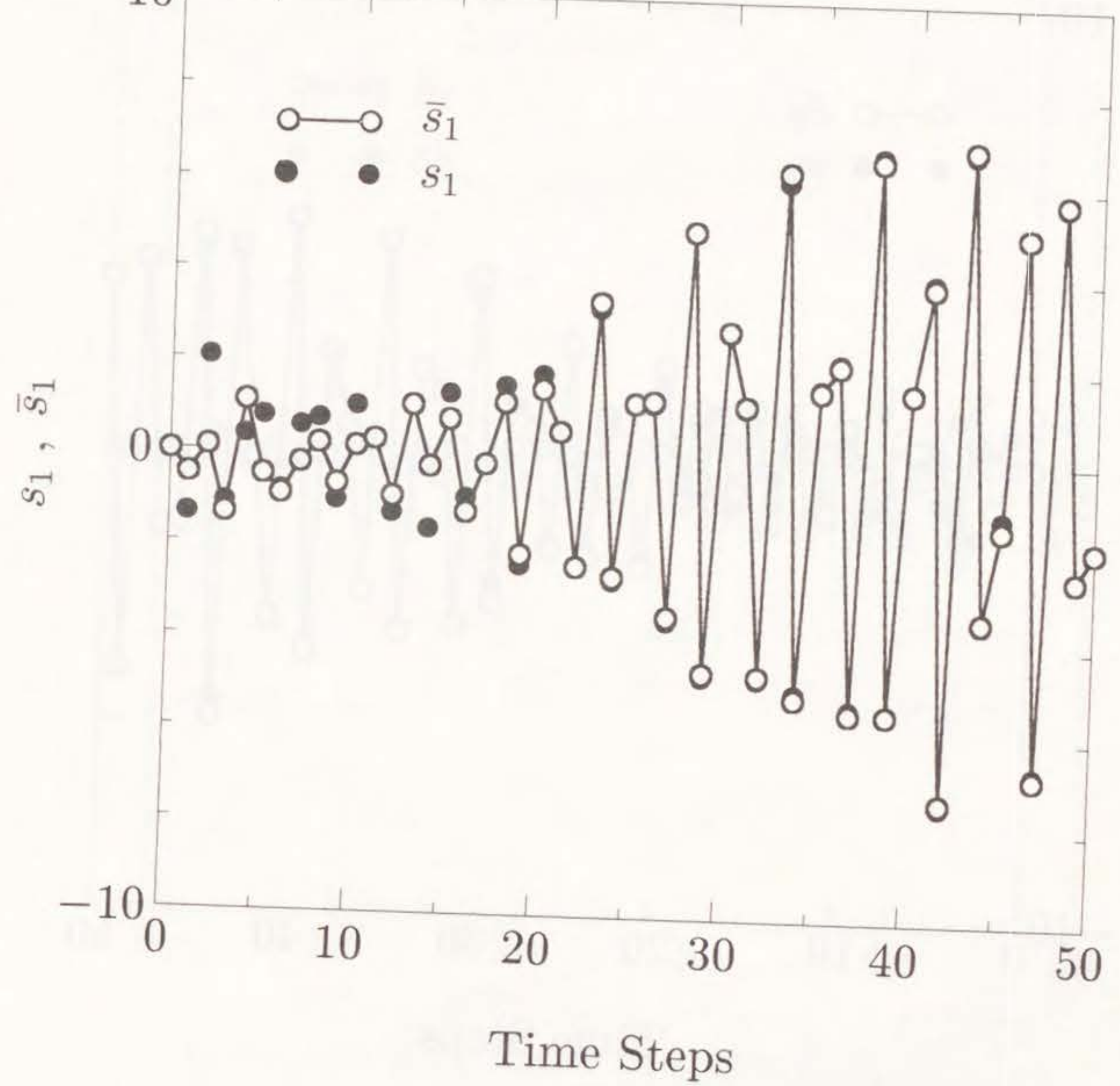

Figure 4.9: Estimation process of the disturbance $s_1(k)$ of $s(k)$(Example 4.1).

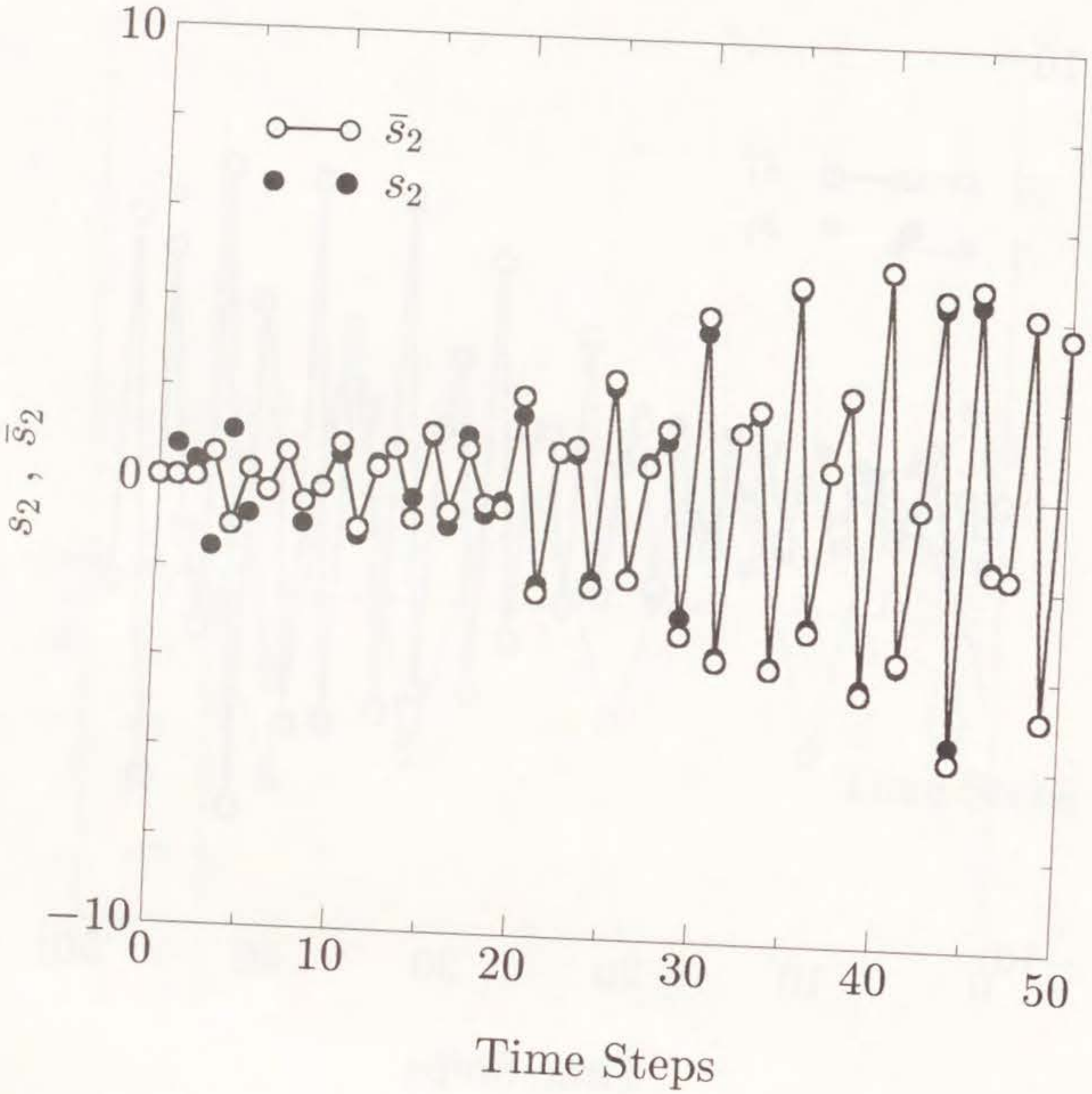

Figure 4.10: Estimation process of the disturbance $s_2(k)$ of $s(k)$(Example 4.1).

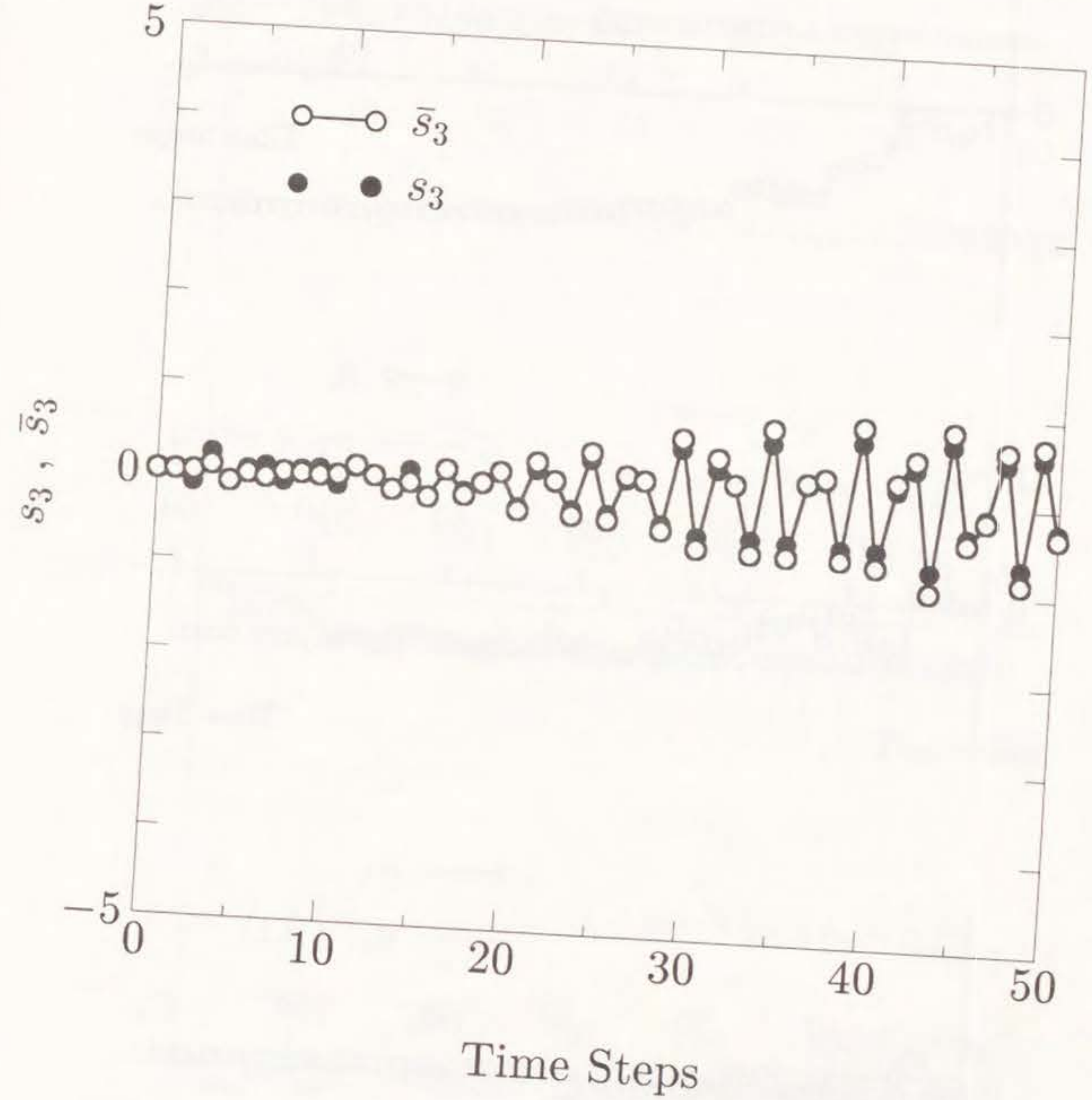

Figure 4.11: Estimation process of the disturbance $s_3(k)$ of $s(k)$(Example 4.1).

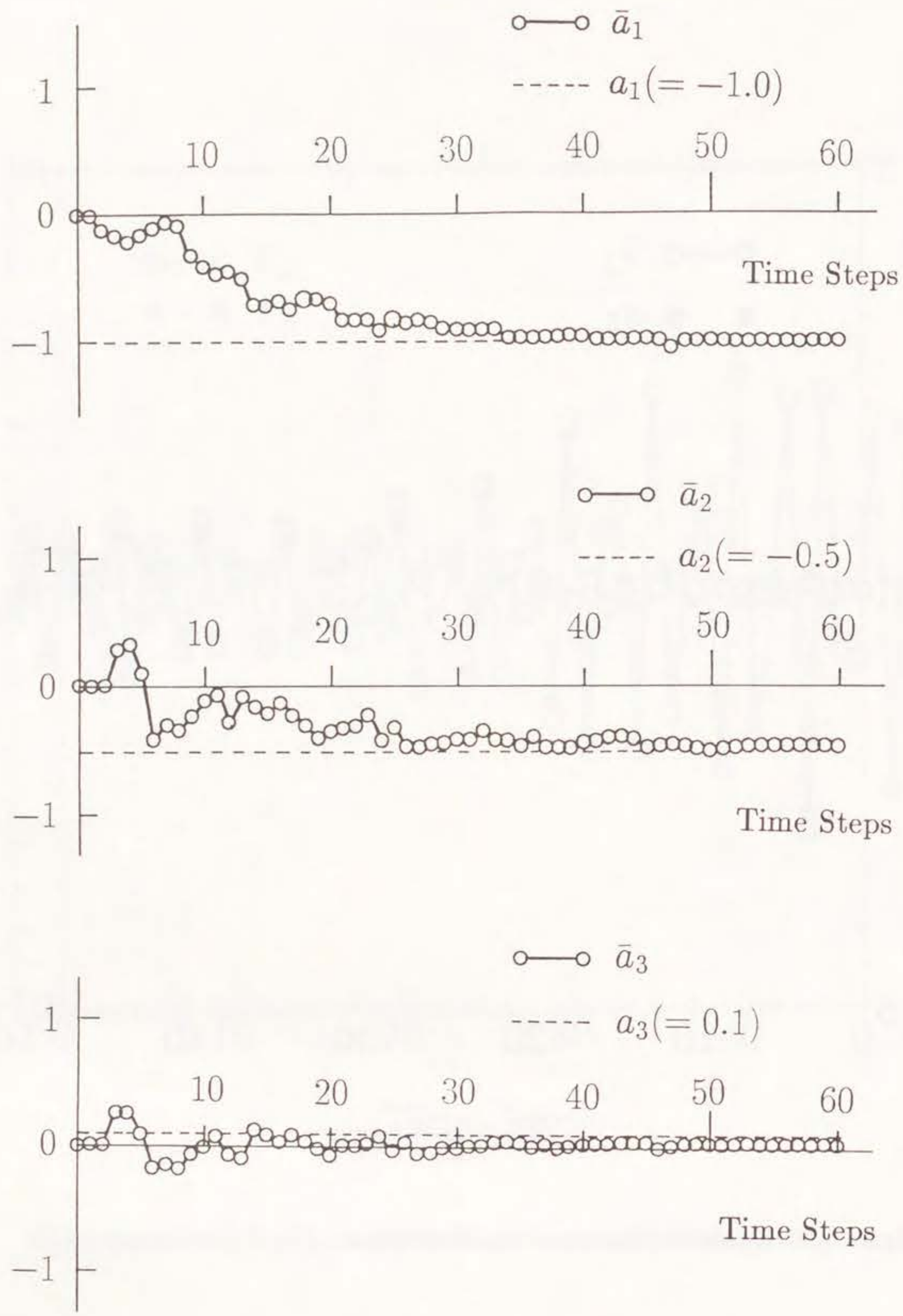

Figure 4.12: Estimation process of the parameter a_s(Example 4.1).

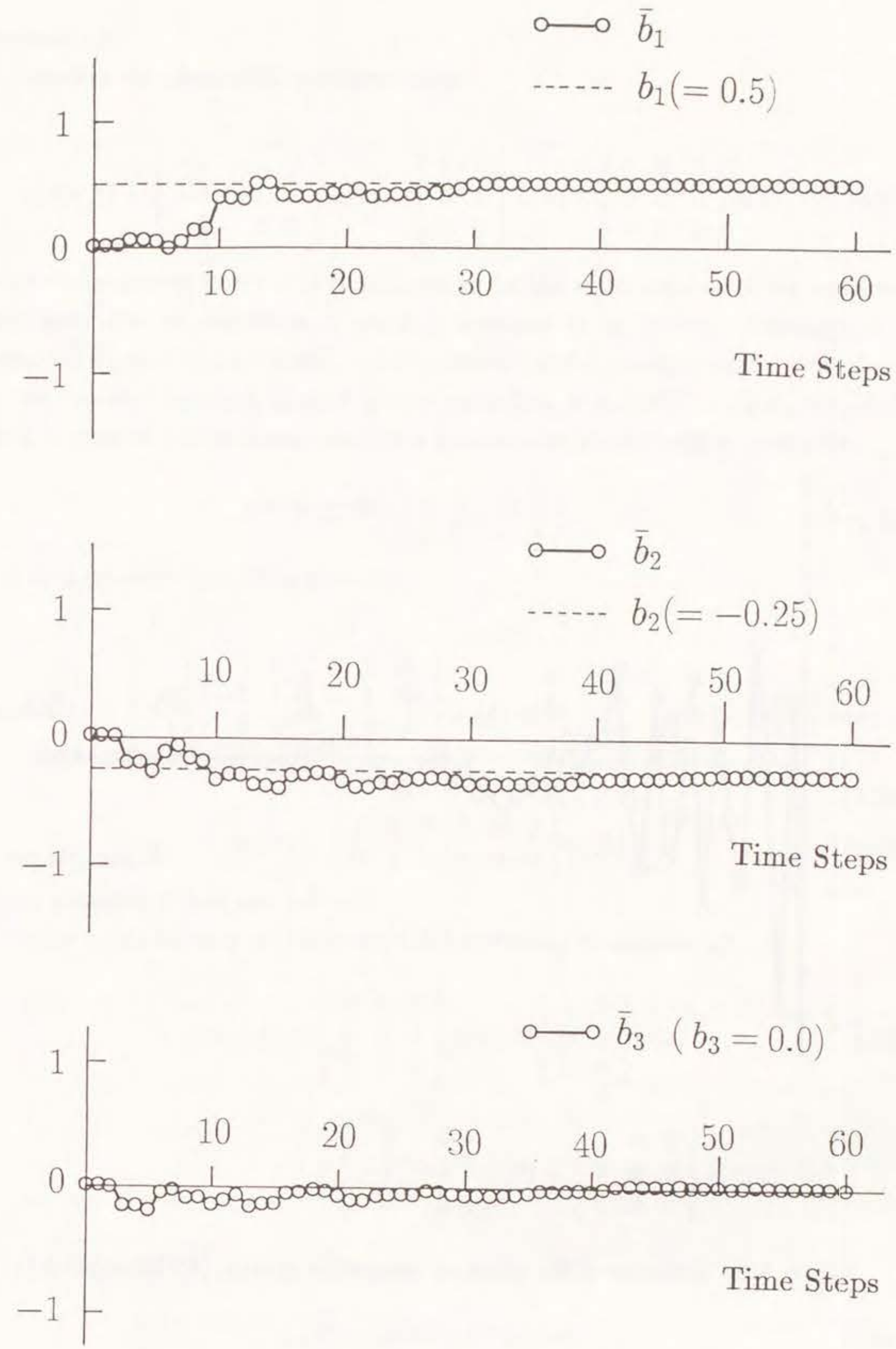

Figure 4.13: Estimation process of the parameter b_s(Example 4.1).

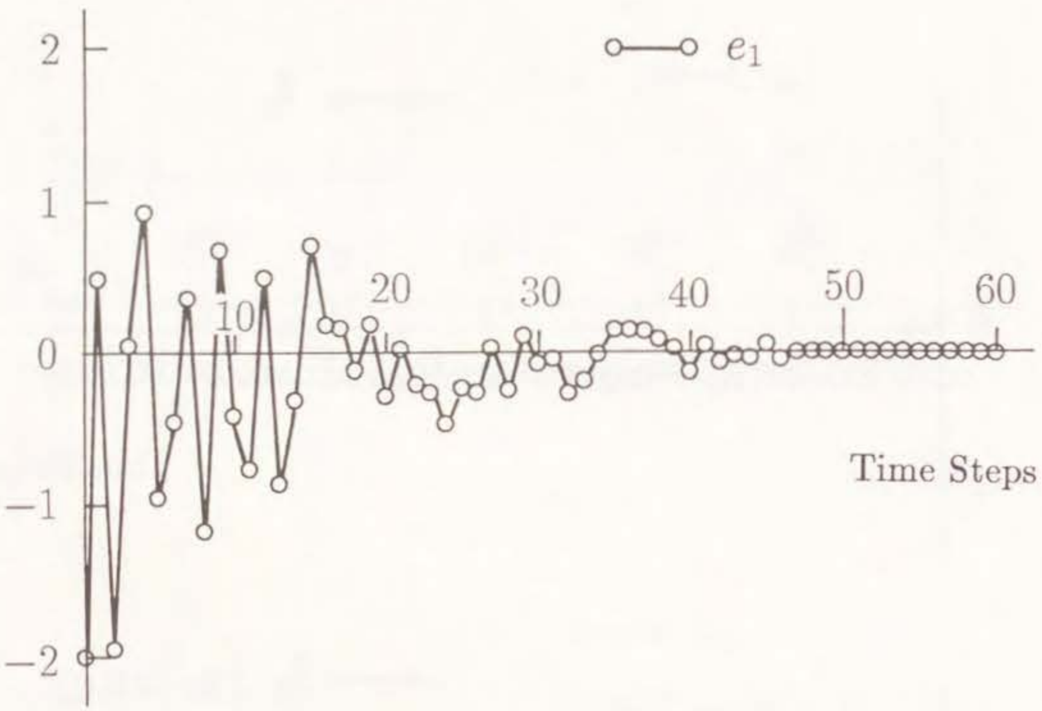

Figure 4.14: Behavior of the adaptive estimation error $e_1(k)$(Example 4.1).

Example 4.2.

Consider the plant with unknown input :

$$x(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} s(k)\,, \qquad (4.89)$$

where the unknown input $s(k)$ is assumed to be the state variable of the unknown input generator of dimension 3 which is assumed to be known. Although A is unstable ($\sigma(A) = \{0.76 \pm 0.86i\,,\ -1.52\}$, where i is the purely imaginary number), we can choose a feedback control gain F such that $\sigma(A+BF) = \{0.0\,,\ 0.0\,,\ 0.0\}$ which is a set of stable closed-loop poles because the plant (4.89) is reachable :

$$F = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & -1 \end{bmatrix}. \qquad (4.90)$$

The augmented system is given by

$$x_e(k+1) = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ \hline \multicolumn{3}{c|}{0_{3\times 3}} & \multicolumn{3}{c}{A_s} \end{array}\right] x_e(k) + \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \hline \multicolumn{2}{c}{0_{3\times 2}} \end{array}\right] u(k) + \left[\begin{array}{c} 0_3 \\ \hline b_s \end{array}\right] w(k)\,, \qquad (4.91)$$

$$y_e(k) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] x_e(k)\,. \qquad (4.92)$$

In this example, C has row full rank.

Define a full order state observer (4.93) according to Lemma 4.2 :

$$\hat{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(k) \qquad (4.93)$$

$$+ \begin{bmatrix} 1 & 0 \\ -2 & -1 \\ 0 & 0 \end{bmatrix} y_e(k) + \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} s(k)\,.$$

Let

$$y_s(k) = [0\,0\,0\,1\,0\,0]\, x_e(k)\,. \qquad (4.94)$$

According to Theorem 4.2, such two dimensional vectors $\tilde{\xi}_1, \tilde{\xi}_2$ which satisfy (4.31) are given by

$$\tilde{\xi}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \tilde{\xi}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4.95}$$

Therefore, using (4.95) and Lemma 4.4, we obtain the linear functional observer of 1 dimension as follows:

$$z(k+1) = \begin{bmatrix} -1 & -1 \end{bmatrix} y_e(k) + \begin{bmatrix} -1 & 0 \end{bmatrix} u(k), \tag{4.96}$$

$$\hat{y}_s(k) = z(k) + [1 \;\; 0] y_e(k). \tag{4.97}$$

Using the same description of the unknown input generator (4.80), the construction of the adaptive state observer is the same as Example 4.1.

Taking, from (4.52),

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

then, the feedback gain F (4.90) satisfies the condition 1 of Theorem 4.3. Using the estimate $\bar{s}(k)$ (in (4.84)) and choosing T which satisfies the condition 2 of Theorem 4.3 as

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \tag{4.98}$$

finally, we obtain the modified control input $u_m(k)$ from (4.55) as

$$u_m(k) = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & -1 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \bar{s}(k). \tag{4.99}$$

Figure 4.15, 4.16 and 4.17 show the behavior of the estimate $\hat{x}(k)$ of the full order observer (4.93) based on the estimated unknown input $\bar{s}(k)$ derived from the adaptive state observer, Figure 4.18 and 4.19 show the behavior of the output $y_o(k)$ of the plant (4.89) based on the proposed controller $u_m(k)$ (given by (4.99)) under unknown input, where the output $y_o(k)$ is forced to zero from the initial state $x_0 = [1 \;\; 1]^T$. Figure 4.20 and Figure 4.21 show the estimation processes of the unknown parameters a_s and b_s in (4.80). In this example, the true values of a_s and b_s are $a_s = [-1 \; -0.5 \; 0.1]^T$ and $b_s = [0.5 \; -0.25 \; 0]^T$. Finally, Figure 4.22 shows the transient response of the error equation (4.86), which shows the adaptive estimation error tends to zero as time-steps.

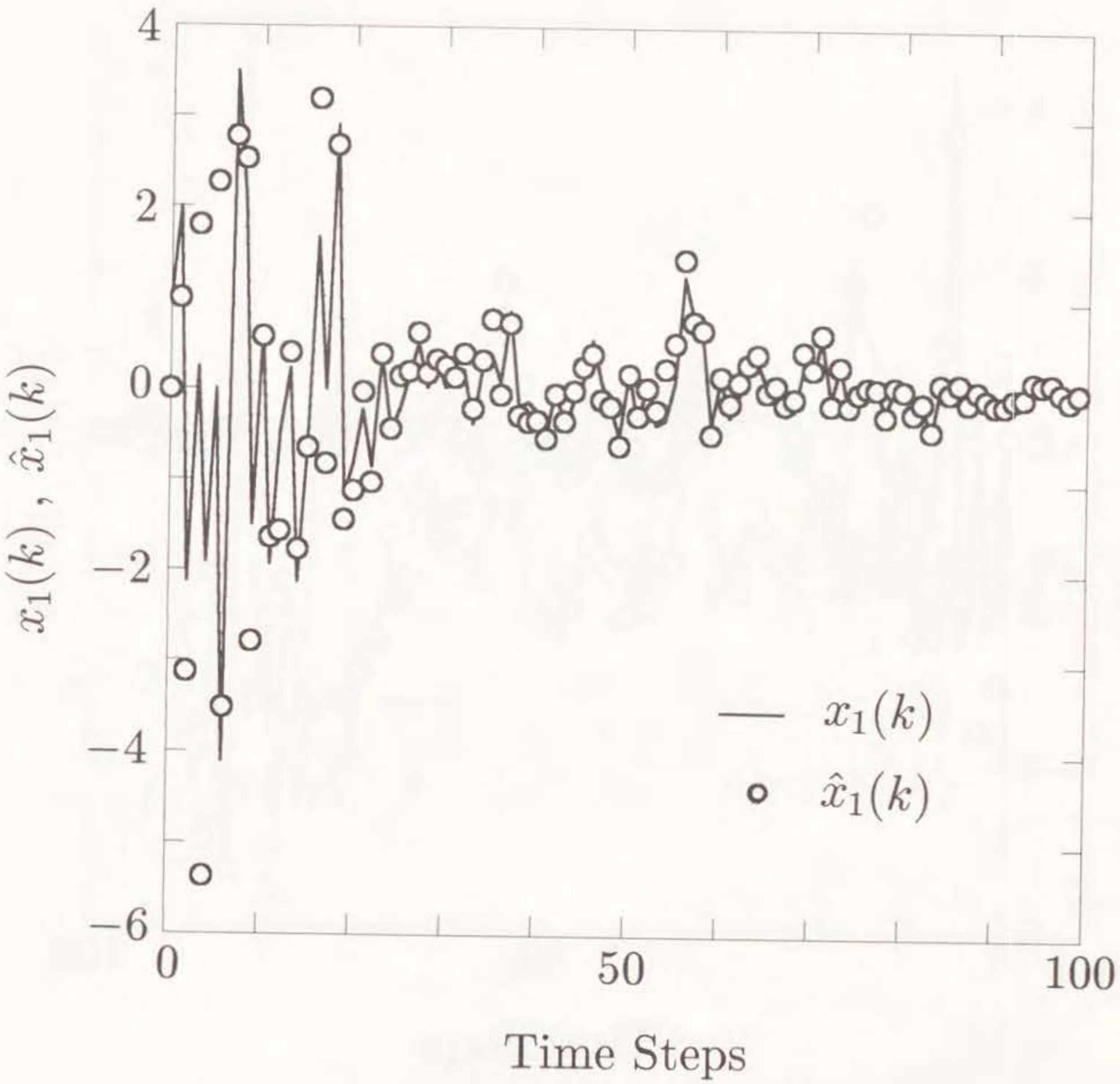

Figure 4.15: Behavior of the estimate $\hat{x}_1(k)$ of $\hat{x}(k)$ based on the estimated unknown input $\bar{s}(k)$(Example 4.2).

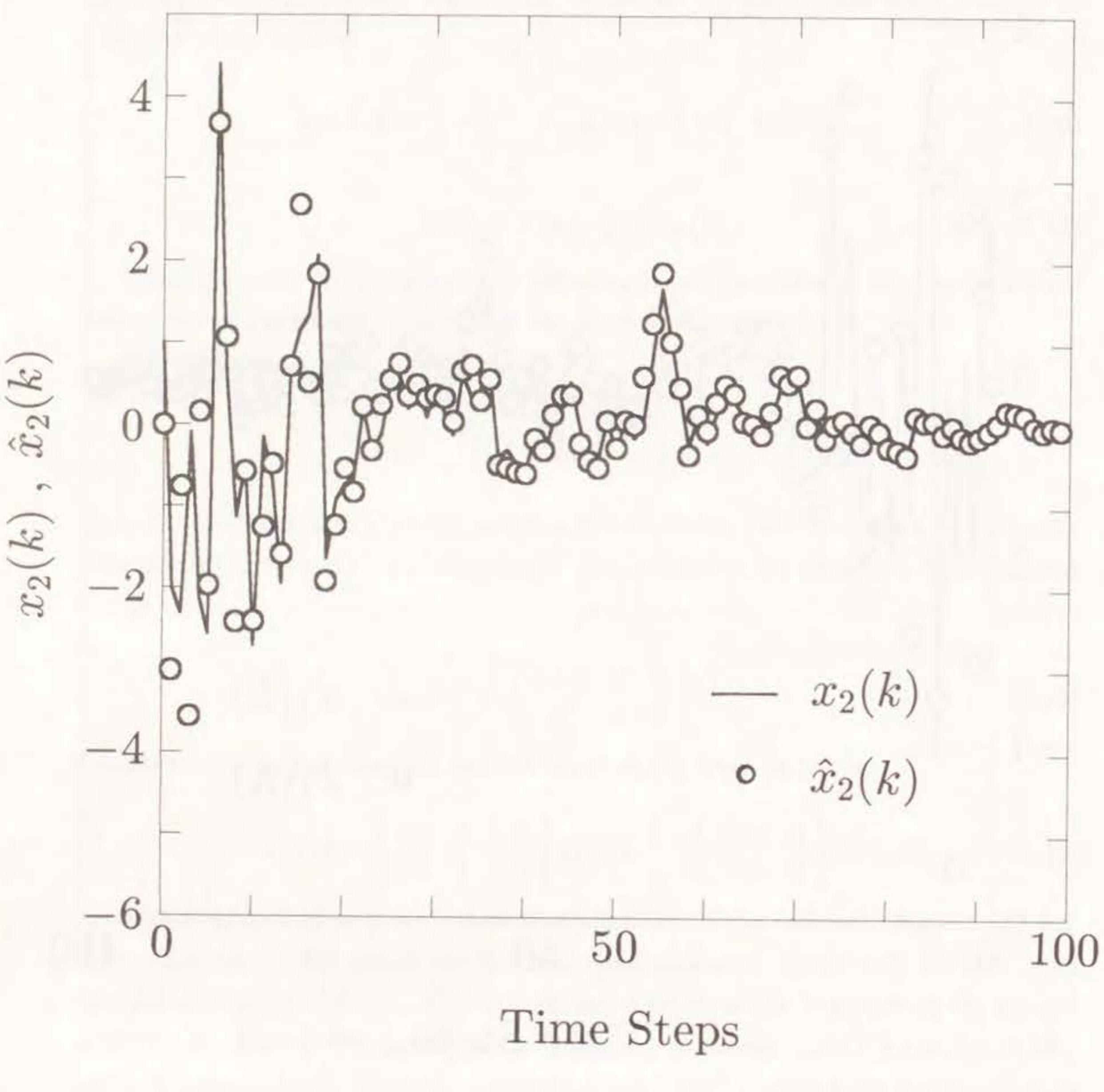

Figure 4.16: Behavior of the estimate $\hat{x}_2(k)$ of $\hat{x}(k)$ based on the estimated unknown input $\bar{s}(k)$(Example 4.2).

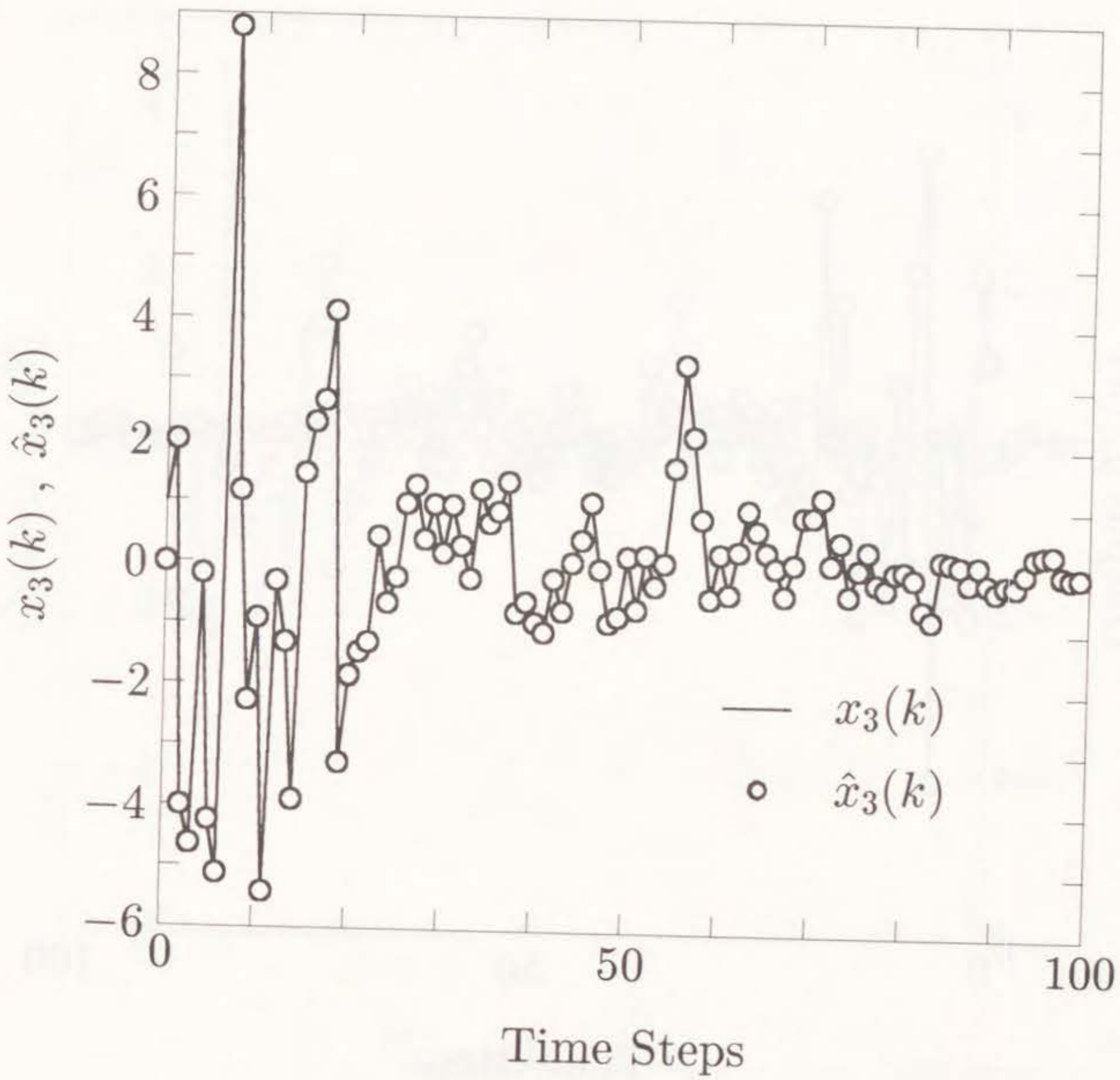

Figure 4.17: Behavior of the estimate $\hat{x}_3(k)$ of $\hat{x}(k)$ based on the estimated unknown input $\bar{s}(k)$(Example 4.2).

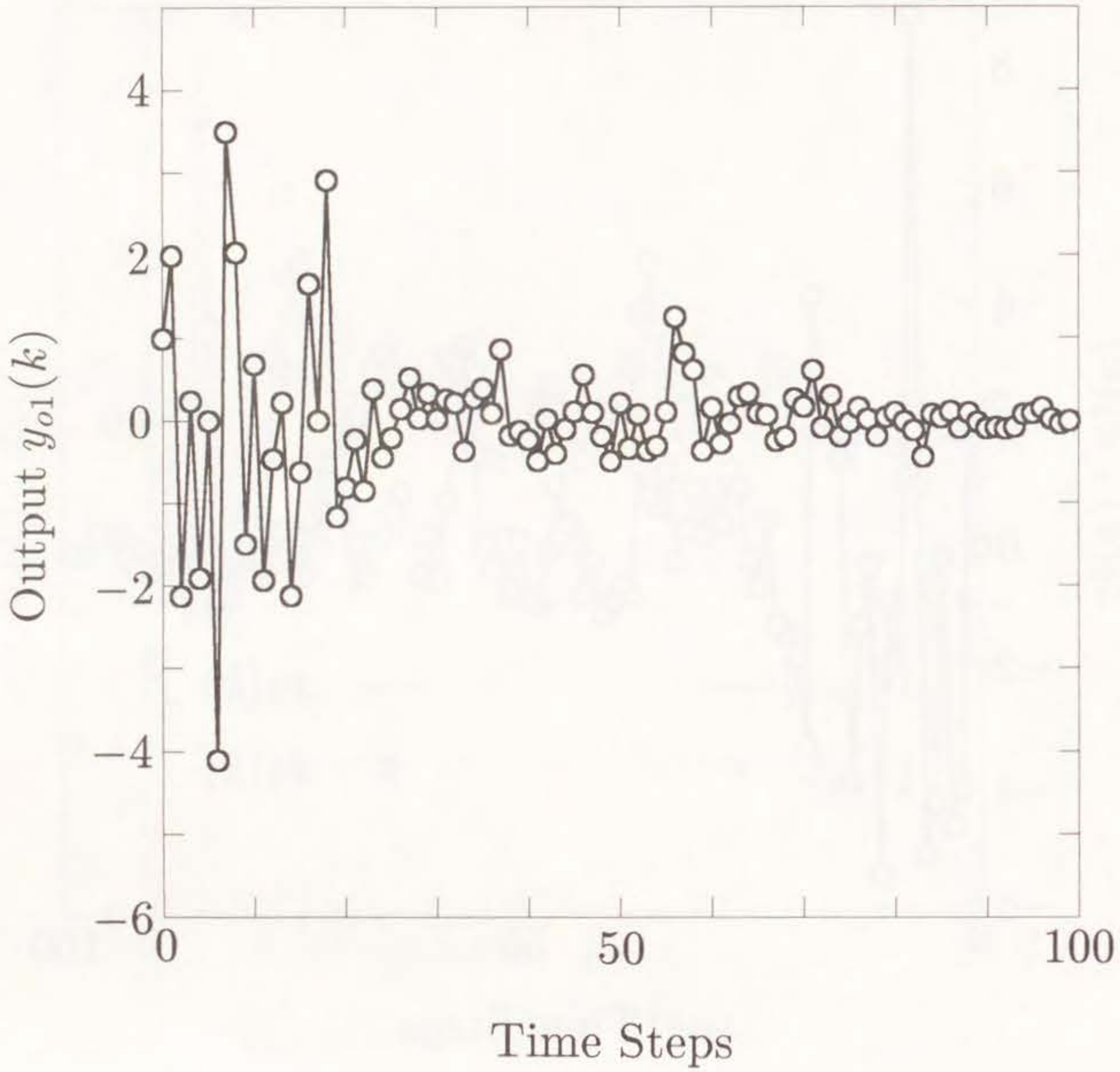

Figure 4.18: Behavior of the first element $y_{o1}(k)$ of the output $y_o(k)$ based on the proposed controller $u_m(k)$(Example 4.2).

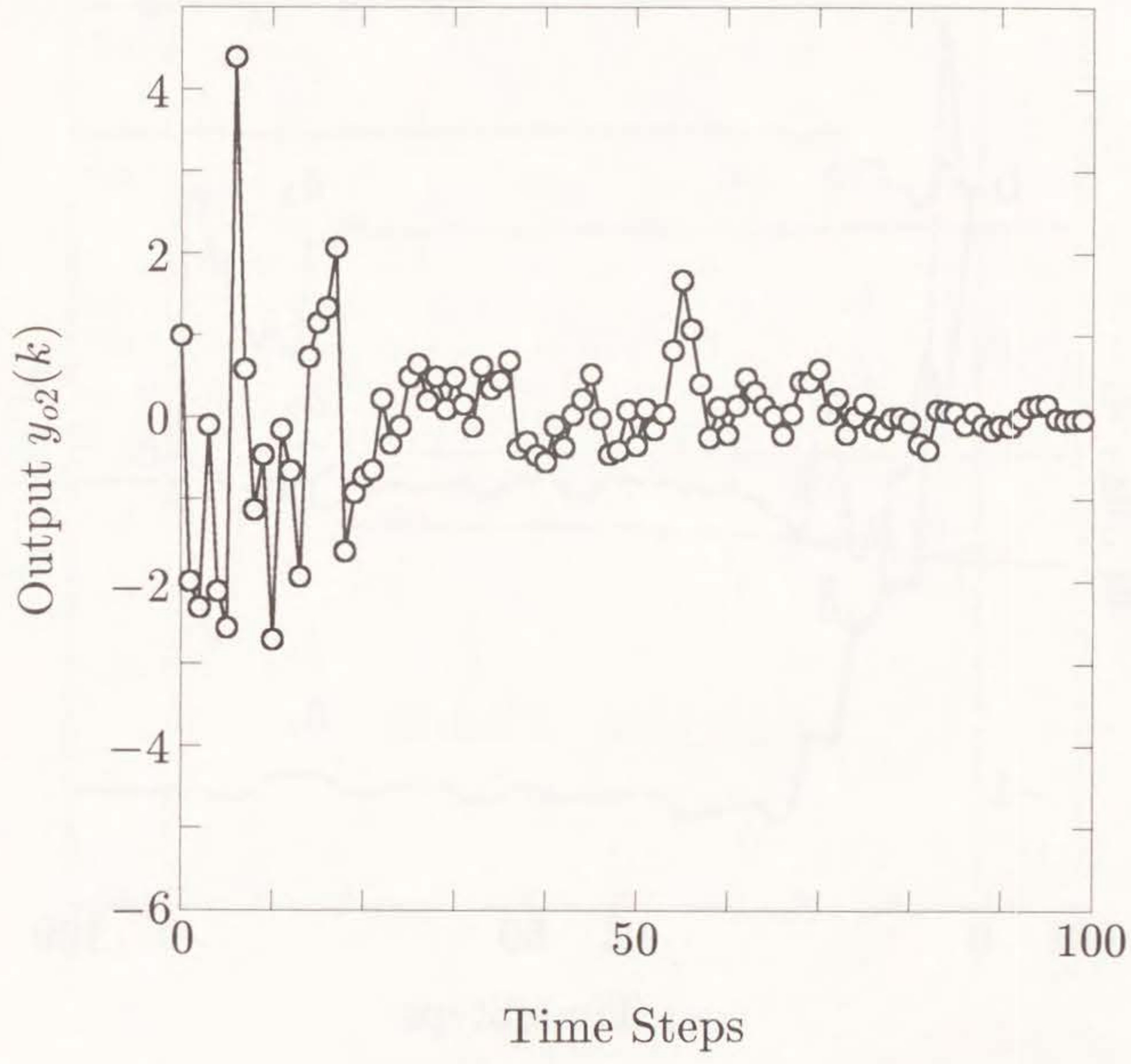

Figure 4.19: Behavior of the second element $y_{o2}(k)$ of the output $y_o(k)$ based on the proposed controller $u_m(k)$(Example 4.2).

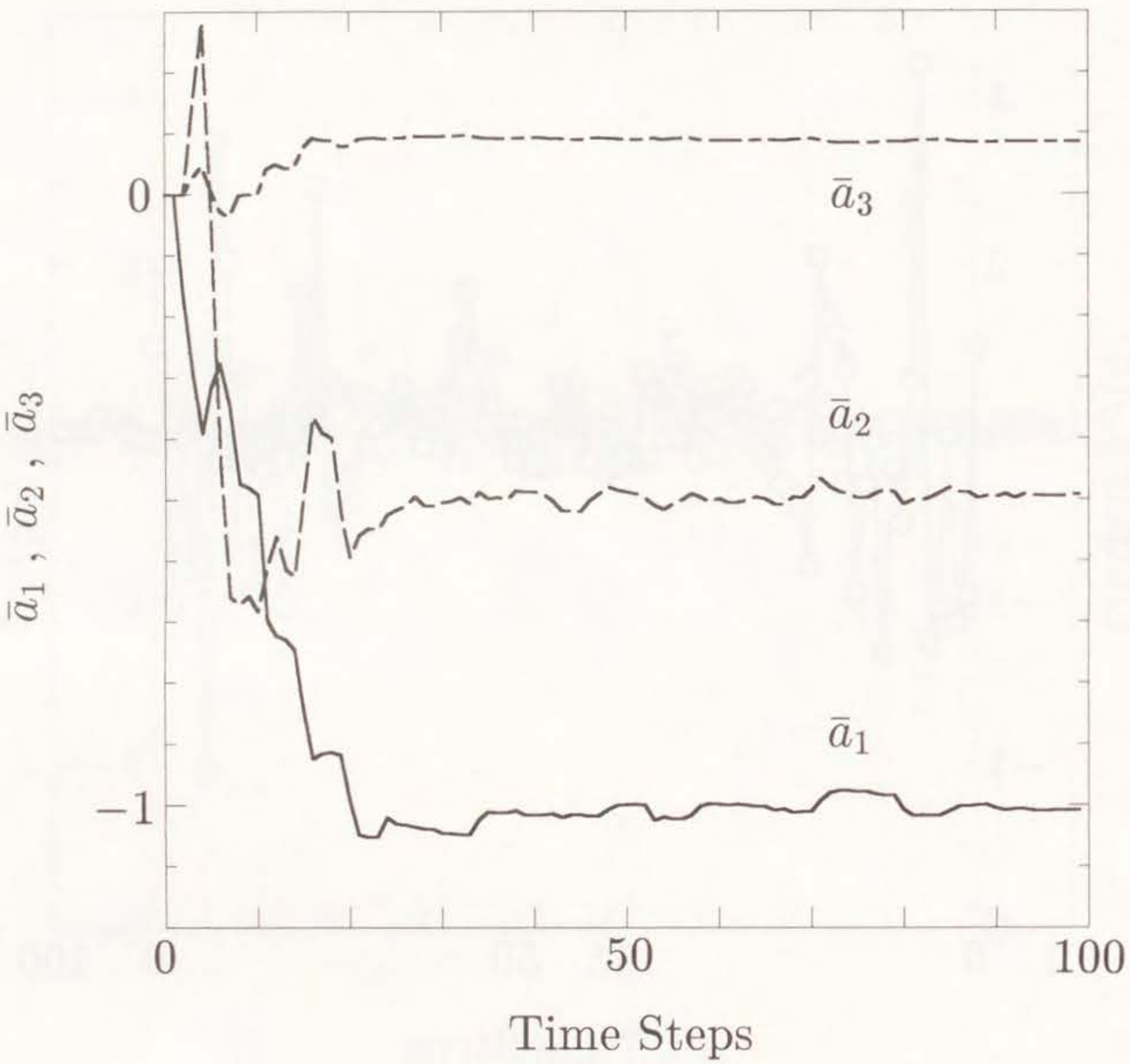

Figure 4.20: Estimation process of the parameter a_s(Example 4.2).

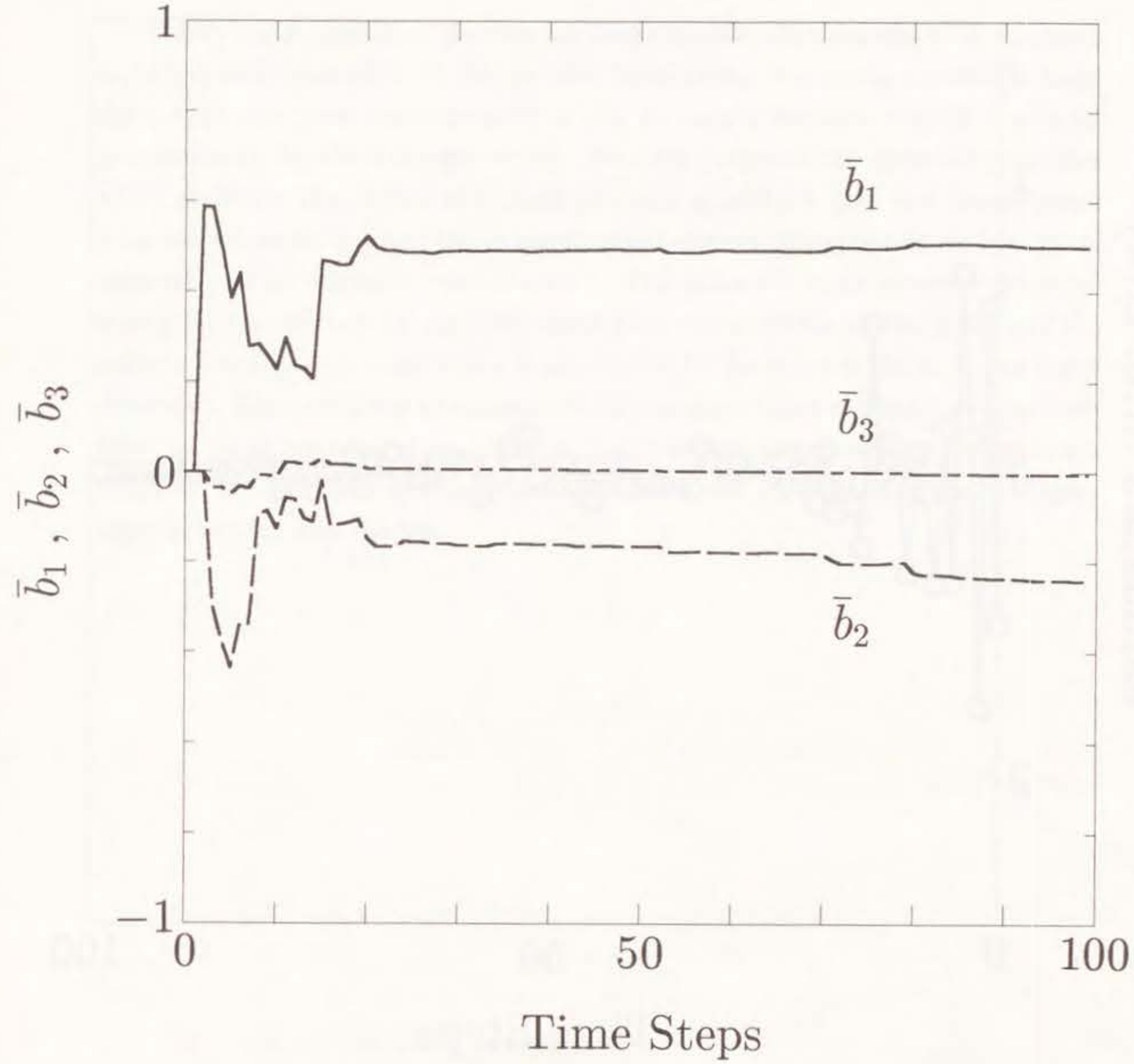

Figure 4.21: Estimation process of the parameter b_s(Example 4.2).

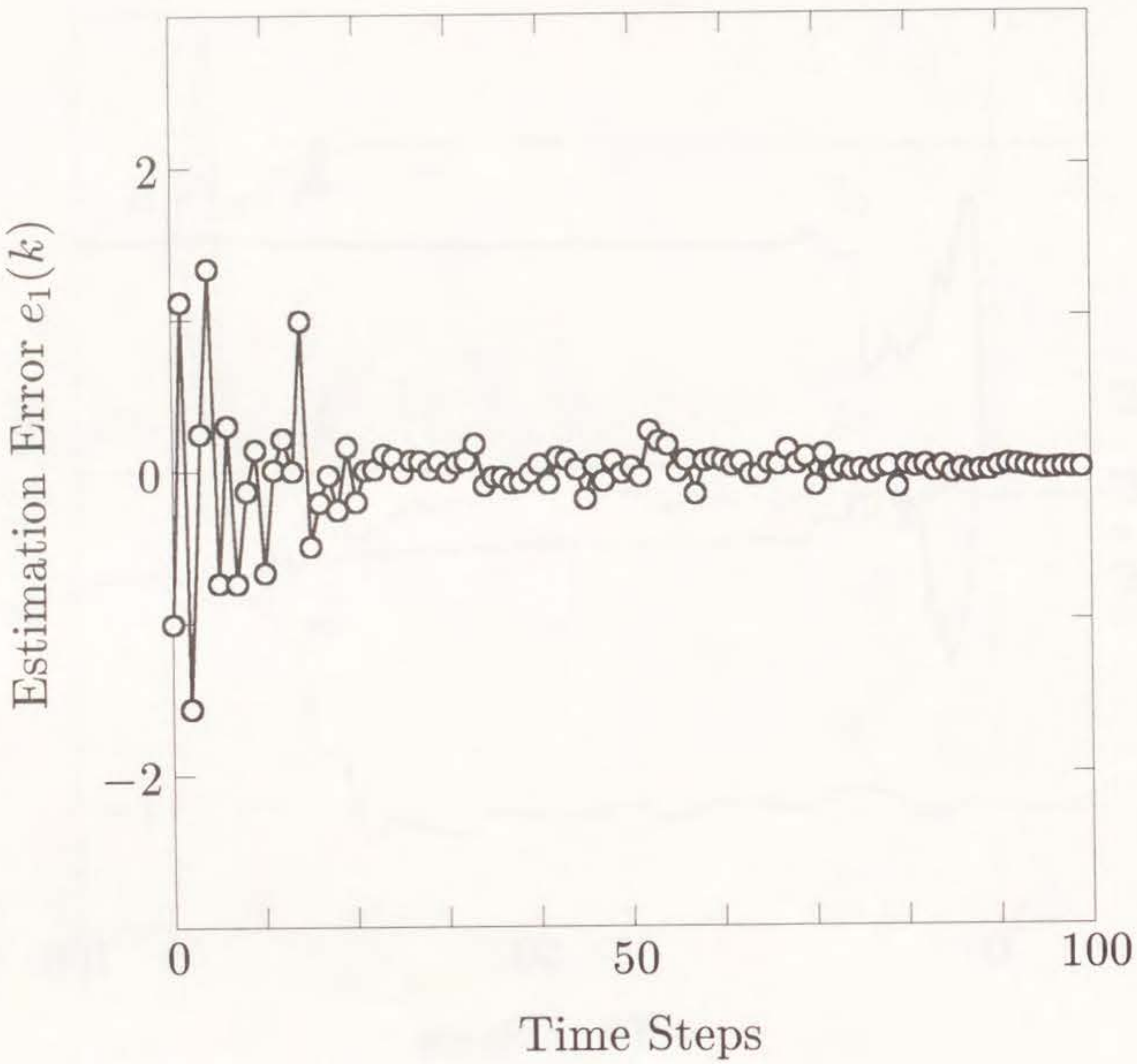

Figure 4.22: Behavior of the adaptive estimation error $e_1(k)$(Example 4.2).

4.7 Conclusion

The dynamic regulation problem for linear discrete-time systems with unknown input has been considered. In this problem formulation, it is hardly possible to make the output of a plant asymptotically stable through a feedback controller without compensating for the unknown input. We have proposed the dynamic controller which stabilizes the output of a plant by using a feedback gain and compensates for unknown input by using the unknown input observer through a linear functional observer and an adaptive state observer. The unknown input observer proposed here gives the estimate of unknown input from the available output $y_e(k)$ and the scalar white Gaussian noise which is assumed to be the input of the unknown input generator. Since unknown parameters of the unknown input generator can be identified by using adaptive state observer, it is possible to compensate for unknown input by the proposed dynamic controller when any parameters of the unknown input generator may change.

Chapter 5

DUAL CONTROLLERS FOR LINEAR DISCRETE-TIME SYSTEMS WITH UNKNOWN INPUT

5.1 Introduction

The controller design method using the adaptive control is one of the most useful design methods in modern control theory. The adaptive controller is a control algorithm which is capable of initially tuning itself and of retuning itself in the event that the process characteristics subsequently change. Although, in this algorithm, the parameter adaptation process must be executed in order to attain the prescribed control action, there are identification errors in the estimated parameters from the beginning to the end of parameter estimation process. The response of the plant output to be controlled is, therefore, disturbed due to such adaptation errors at least until the adaptive algorithm converges successfully.

In this chapter, we are concerned with the control problem of uncertain linear systems described by difference equations which contain unknown parameters. For these systems, the dual controller is proposed which implements efficient control actions by using two kinds of controllers – the stochastic sub-optimal regulator and the deterministic adaptive regulator – and switching them effectively.

Figure 5.1 illustrates the control system with the dual controller. The controllers used here are designed for the augmented system composed of the plant to be controlled and the unknown input generator (see Chapter 4). Because the unknown input generator is assumed to have a linear system structure with random input, the optimal regulator is derived for a stochastic problem formulation. The stochastic discrete-time linear optimal regulator with the quadratic criterion in a finite time interval to be minimized is obtained by the form of a feedback controller with the time-variant gain and the estimate of the state variable [7, 46]. On the other hand, the present adaptive regulator uses the time-invariant feedback gain.

In the control action using the dual controller, when the plant begins to work, the optimal regulator begins to control the plant first and the parameter estimation process begins to work at the same time. Until the estimation error decreases in the adaptive process, the optimal regulator continues to control the plant. When the estimation error becomes sufficiently small, the controller is switched and the adaptive regulator begins to control the plant instead of the optimal regulator. When the uncertainties becomes large as the parameters of the unknown input generator change during control process, the adaptive regulator is replaced by the optimal regulator again until the next estimation process converges.

There are mainly two reasons why these two controllers are used together for a single plant. One is the adaptive control algorithm inherently includes estimation errors in transient responses. Another is the stochastic optimal regulator gives the optimal response so that the performance criterion is minimized, but the responses using the stochastic optimal regulator include larger errors than those using the adaptive regulator with unbiased estimates.

The rest of this chapter is organized as follows. In Section 5.2, the problem formulation and definitions are given. The stochastic sub-optimal regulator composed of the optimal gain and the sub-optimal estimate is derived in Section 5.3. In Section 5.4, we consider that the switching scheme between the adaptive controller (Chapter 4) and the stochastic sub-optimal regulator (Section 5.3). A numerical example with simulation results is given in Section 5.5.

5.2 Problem Statement and Preliminaries

The control system to be considered here is the same as the system of Chapter 4. The augmented system composed of the plant to be controlled and the unknown input generator is given by the same form as (4.3)–(4.4).

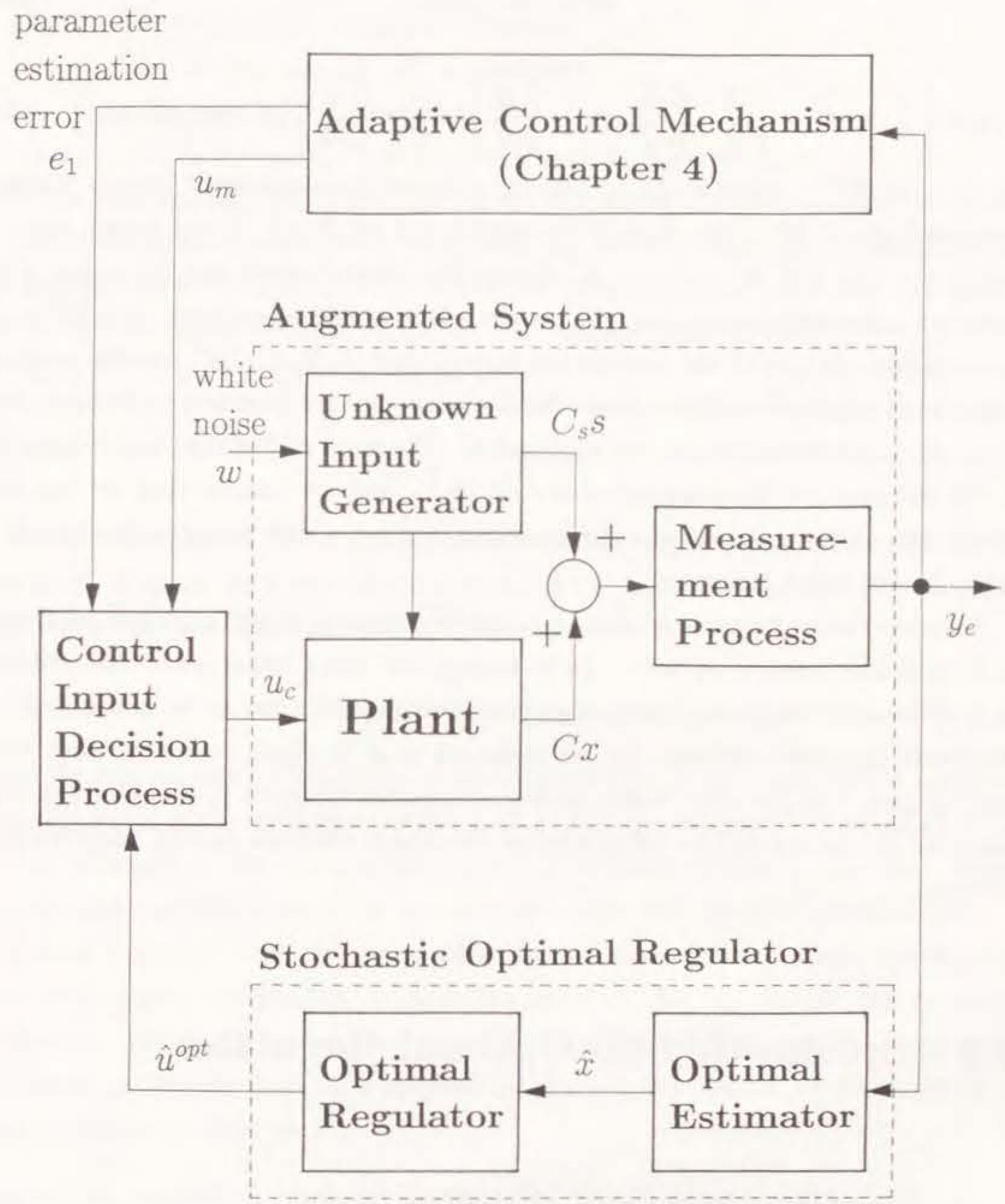

Figure 5.1: Block diagram of the dual control system.

$$x_e(k+1) = A_e x_e(k) + B_e u(k) + b_{se} w(k), \quad (5.1)$$

$$y_e(k) = C_e x_e(k)\,, \quad (5.2)$$

$$x_e = \begin{bmatrix} x \\ s \end{bmatrix},\ A_e = \begin{bmatrix} A & E \\ 0 & A_s \end{bmatrix},\ B_e = \begin{bmatrix} B \\ 0 \end{bmatrix},\ b_{se} = \begin{bmatrix} 0 \\ b_s \end{bmatrix},\ C_e = \begin{bmatrix} C & C_s \end{bmatrix}, \quad (5.3)$$

where $x_e \in \mathbf{R}^{n+n_s}$ denotes the (extended) state of the augmented system which is composed of $x \in \mathbf{R}^n$ (: the state of the plant) and $s \in \mathbf{R}^{n_s}$ (: the unknown input to the plant), and $u \in \mathbf{R}^r$ and $w \in \mathbf{R}^1$ denote the control input and the scalar white Gaussian noise with zero mean and a unity covariance, respectively. $y_e \in \mathbf{R}^p$ is the measurement output of the augmented system, and A, B, E, C, C_s are the constant matrices of appropriate dimensions whose parameters are assumed to be given, and A_s, b_s are matrices with unknown parameters. The state $x(\cdot)$ of the plant is regarded as the sub-state of the augmented system (5.1), and we assume that we can only obtain the output $y_e(\cdot)$, as the measurement output which includes the output of the unknown input generator.

Because the augmented system (5.1) has the random signal w, it can be treated as a stochastic control system. As is known, for some linear quadratic criterion with symmetric weighting matrices as the performance index to be minimized, the stochastic optimal regulator can be obtained as a feedback regulator with time-varying gain. On the other hand, as is mentioned in Chapter 4, the adaptive regulator can be constructed by using a linear functional observer and an adaptive state observer.

5.3 Stochastic Sub-Optimal Regulator

In this section, the stochastic sub-optimal regulator is derived for the augmented system (5.1),(5.2). Now consider the following performance criterion to be minimized :

$$J(u) = E[\,x_e^T(T) Q_0 x_e(T) + \sum_{k=0}^{T-1} \{\, y_e^T(k) Q y_e(k) + u^T(k) R_u u(k)\,\}\,]\,, \quad (5.4)$$

where $E[\cdot]$ denotes the expectation operator. The weighting matrices Q_0, Q are chosen to be positive semi-definite symmetric, and R_u is chosen to be positive definite symmetric.

Now we make the following assumptions as in Chapter 4 :

(A.1) The triplet $(C\,, A\,, B)$ is complete.
That is, the pair $(A\,, B)$ is reachable,
and the pair $(C\,, A)$ is observable.

(A.2) C has full rank, $C_s \neq 0$, $A_s \neq 0$ and A_s is stable.

From the assumptions **(A.1)** and **(A.2)**, we can derive the observability condition for the augmented system (5.1), (5.2) with $b_{se} \equiv 0$ (Lemma 4.1). Because any information about the system structure of the unknown input generator can not be used in the stochastic optimal regulator, it is further assumed that

(A.3) The matrix Q_0 is chosen so that

$$Q_0 = \begin{bmatrix} Q_{01} & 0 \\ 0 & 0 \end{bmatrix}. \quad (5.5)$$

For the augmented system (5.1), (5.2) with the performance index (5.4) to be minimized, if there are given all parameters of the matrices A_e, B_e, b_{se}, C_e, it is well-known that the optimal input $\hat{u}^{opt}(k)$ is given by

$$\hat{u}^{opt}(k) = [\,F(k)\ \ G(k)\,]\,\hat{x}_e(k)\,, \quad (5.6)$$

where the gain matrix $[\,F(k)\ \ G(k)\,]$ is the optimal gain in the sense that it gives the minimum value of the performance index (5.4) and the estimated state $\hat{x}_e(k)$ is the optimal estimate in the sense of the minimum variance estimate. In other words, the separation principle holds for the stochastic optimal regulator problem [87].

But on the other hand, in the problem formulation of this chapter, the optimal input (5.6) cannot be derived, because the matrices A_e, b_{se} include the unknown parameters. Although the optimal gain $[\,F(k)\ \ G(k)\,]$ of (5.6) can be obtained as mentioned in Section 5.3.1, the optimal estimate $\hat{x}_e(k)$ cannot be constructed by using an ordinary observer :

$$\hat{x}_e(k+1) = A_e \hat{x}_e(k) - \Gamma_e(k)\{y_e(k) - C\hat{x}_e(k)\} + B_e \hat{u}^{opt}(k)\,, \quad (5.7)$$

where $\Gamma_e(k)$ is the adjusting matrices, because the augmented system matrix A_e, which includes the unknown matrix A_s, cannot be used in the observer (5.7).

Instead of the stochastic optimal regulator problem using the optimal input (5.6), the problem is considered in which the sub-optimal input (5.8) is introduced to the augmented system (5.1),(5.2) with the performance index (5.4) to be minimized :

$$u^+(k) = [\,F(k) \;\; G(k)\,]\, x_e^+(k)\,, \tag{5.8}$$

where the gain matrix $[\,F(k) \;\; G(k)\,]$ is the optimal gain in the sense that it gives the minimum value of the performance index (5.4) and the estimated state $x_e^+(k)$ is the sub-optimal estimated state in the sense that it gives the minimum norm of x_e with the least squares of $(y_e - C_e x_e)$ which is derived from the measurement output (5.2).

5.3.1 Optimal gain

The optimal gain of (5.8) is derived according to the following theorem.

Theorem 5.1 *The optimal gain $[\,F(k) \;\; G(k)\,]$ of (5.8) is derived from the following recursive equations.*

$$\begin{aligned} F(k) &= -[B^T\Phi_1(k+1)B + R_u]^{-1}B^T\Phi_1(k+1)A\,, \\ G(k) &= -[B^T\Phi_1(k+1)B + R_u]^{-1}B^T\Phi_1(k+1)E\,, \end{aligned} \tag{5.9}$$

where $\Phi_1(k)$ $(k = T, T-1, \ldots, 1, 0)$ is the symmetric matrix given by the following recursive matrix equations.

$$\begin{aligned} \Phi_1(k) &= A^T\Phi_1(k+1)A + C^TQC \\ &\quad - A^T\Phi_1(k+1)B[B^T\Phi_1(k+1)B + R_u]^{-1}B^T\Phi_1(k+1)A\,, \\ 0 &= A^T\Phi_1(k+1)E + C^TQC_s \\ &\quad - A^T\Phi_1(k+1)B[B^T\Phi_1(k+1)B + R_u]^{-1}B^T\Phi_1(k+1)E\,, \\ 0 &= E^T\Phi_1(k+1)E + C_s^TQC_s \\ &\quad - E^T\Phi_1(k+1)B[B^T\Phi_1(k+1)B + R_u]^{-1}B^T\Phi_1(k+1)E\,, \\ \Phi_1(T) &= Q_{01}\,. \end{aligned} \tag{5.10}$$

Proof. For the time being, assume that the augmented state $x_e(k)$ can be measured directly and there is no disturbance. Then, for the optimal regulator problem of the augmented control system (5.1), (5.2) with (5.5), which results in the deterministic optimal regulator problem, the optimal control input $u^*(k)$ is given by the state feedback with time-varying gain $F_e(k)$:

$$u^*(k) = F_e(k)x_e(k)\,, \tag{5.11}$$

$$F_e(k) = -[B_e^T\Phi_e(k+1)B_e + R_u]^{-1}B_e^T\Phi_e(k+1)A_e\,, \tag{5.12}$$

where $\Phi_e(k)$ $(k = T, T-1, \ldots, 1, 0)$ are calculated from the following recursive matrix equation.

$$\begin{aligned} \Phi_e(k) &= A_e^T\Phi_e(k+1)A_e + C_e^TQC_e \\ &\quad - A_e^T\Phi_e(k+1)B_e[B_e^T\Phi_e(k+1)B_e + R_u]^{-1}B_e^T\Phi_e(k+1)A_e\,, \\ \Phi_e(T) &= Q_0\,. \end{aligned} \tag{5.13}$$

Because the matrix A_e includes unknown parameters A_s which are the system matrix of unknown input generator, the matrix $\Phi_e(k)$ can not be calculated from (5.13) and therefore the optimal gain $F_e(k)$ can not be obtained either. By using the matrix Q_0 of the assumption **(A.3)** and from (5.3), the equation (5.13) at $k = T-1$ can be decomposed into the following equations.

$$\begin{aligned} \Phi_1(T-1) &= A^TQ_{01}A + C^TQC - A^TQ_{01}B\{\Delta(Q_{01})\}^{-1}B^TQ_{01}A\,, \\ \Phi_{12}(T-1) &= A^TQ_{01}E + C^TQC_s - A^TQ_{01}B\{\Delta(Q_{01})\}^{-1}B^TQ_{01}E\,, \\ \Phi_2(T-1) &= E^TQ_{01}E + C_s^TQC_s - E^TQ_{01}B\{\Delta(Q_{01})\}^{-1}B^TQ_{01}E\,, \end{aligned} \tag{5.14}$$

where

$$\Phi_e(T-1) = \begin{bmatrix} \Phi_1(T-1) & \Phi_{12}(T-1) \\ \Phi_{12}(T-1)^T & \Phi_2(T-1) \end{bmatrix},$$

and $\Delta(Q_{01}) := B^TQ_{01}B + R_u$.

If we choose that $\Phi_2(T-1) \neq 0$ or $\Phi_{12}(T-1) \neq 0$, then the symmetric matrix $\Phi_e(T-2)$ can not be obtained from (5.13) since we have no information about unknown parameters A_s. Therefore, we must set that $\Phi_2(T-1) = 0$ and $\Phi_{12}(T-1) = 0$, and thus these conditions lead to (5.10) at $k = T-1$. Through the same discussions, it is concluded that the symmetric matrix $\Phi_1(k)$ $(k = T, T-1, \ldots, 1, 0)$ must satisfy (5.10).

Although Theorem 5.1 guarantees the existence of the optimal feedback gain, it is a tough problem that the symmetric matrix $\Phi_1(k)$ must be chosen so that the recursive conditions (5.10) are satisfied.

Now we further put some additional assumptions.

(A.4) $\operatorname{Im} C$ and $\operatorname{Im} C_s$ are independent.

(A.5) The weighting symmetric matrix Q is assumed to be chosen so that

$$\operatorname{Im} C_s \subset \operatorname{Ker} Q\,.$$

Under the assumptions **(A.4)** and **(A.5)**, we can obtain the following corollary, which gives a less severe condition for obtaining the optimal gain.

Corollary 5.1 *Assume that the assumptions* **(A.1)**–**(A.5)** *hold. Then, the symmetric matrix* $\Phi_1(k)$ $(k = T, T-1, \ldots, 1, 0)$ *of the optimal gain (5.9) is obtained from the following recursive matrix equations.*

$$
\begin{aligned}
\Phi_1(k) &= A^T\Phi_1(k+1)A + C^TQC \\
&\quad - A^T\Phi_1(k+1)B[B^T\Phi_1(k+1)B + R_u]^{-1}B^T\Phi_1(k+1)A\,, \\
0 &= A^T\Phi_1(k+1)E \\
&\quad - A^T\Phi_1(k+1)B[B^T\Phi_1(k+1)B + R_u]^{-1}B^T\Phi_1(k+1)E\,, \\
0 &= E^T\Phi_1(k+1)E \\
&\quad - E^T\Phi_1(k+1)B[B^T\Phi_1(k+1)B + R_u]^{-1}B^T\Phi_1(k+1)E\,, \\
\Phi_1(T) &= Q_{01}\,.
\end{aligned}
\tag{5.15}
$$

Proof. From the assumption **(A.5)**, we obtain that

$$C^TQC_s = 0\,,\ C_s^TQC = 0\,,\ C_s^TQC_s = 0\,.$$

Therefore, (5.10) leads to (5.15).

Corollary 5.2 *Assume that the assumptions* **(A.1)**–**(A.5)** *hold. Moreover, assume that the weighting matrix* R_u *is chosen so that* $R_u \equiv 0$ *in (5.4). Then, the optimal gain* $[\,F(k)\ \ G(k)\,]$ *of (5.8) is derived from the following recursive equations.*

$$
\begin{aligned}
F(k) &= -\{B^T\Phi_1(k+1)B\}^+B^T\Phi_1(k+1)A\,, \\
G(k) &= -\{B^T\Phi_1(k+1)B\}^+B^T\Phi_1(k+1)E\,.
\end{aligned}
\tag{5.16}
$$

Also the symmetric matrix $\Phi_1(k)$ $(k = T, T-1, \ldots, 1, 0)$ *of the optimal gain (5.16) is obtained from the following recursive matrix equation,*

$$
\begin{aligned}
\Phi_1(k) &= A^T\Phi_1(k+1)A + C^TQC \\
&\quad - A^T\Phi_1(k+1)B[B^T\Phi_1(k+1)B]^+B^T\Phi_1(k+1)A\,, \\
\Phi_1(T) &= Q_{01}\,,
\end{aligned}
\tag{5.17}
$$

if the matrix $D(k)B$ $(k = T, T-1, \ldots, 1, 0)$ *is nonsingular, where the matrix* $D(k)$ *is the nonzero matrix so that*

$$\Phi_1(k) = D^T(k)D(k)\,,\ (k = T, T-1, \ldots, 1, 0)\,. \tag{5.18}$$

Proof. Assume that the matrix $D(k)$ is chosen so that (5.18) is satisfied. By choosing $R_u \equiv 0$, the right-hand side of the second equation of (5.15) is given by

$$
\begin{aligned}
& A^T[D^T(k+1)D(k+1) - D^T(k+1)D(k+1)B \\
&\quad \times\{B^TD^T(k+1)D(k+1)B\}^+B^TD^T(k+1)D(k+1)]E \\
&= A^TD^T(k+1)[I - D(k+1)B \\
&\quad \times\{B^TD^T(k+1)D(k+1)B\}^+B^TD^T(k+1)]D(k+1)E \\
&= A^TD^T(k+1)[I - D(k+1)B\{D(k+1)B\}^+]D(k+1)E\,,
\end{aligned}
\tag{5.19}
$$

where the relation

$$\{B^TD^T(k+1)D(k+1)B\}^+B^TD^T(k+1) = \{D(k+1)B\}^+$$

is used (see (1.9) of Chapter 1). Similarly, the right-hand side of the third equation of (5.15) is given by

$$E^TD^T(k+1)[I - D(k+1)B\{D(k+1)B\}^+]D(k+1)E\,. \tag{5.20}$$

If $D(k)B$ $(k = T, T-1, \ldots, 1, 0)$ is nonsingular, (5.19) and (5.20) are both equal to zero, therefore, the second and third equation of (5.15) are always satisfied. Therefore, the symmetric matrix $\Phi_1(k)$ is obtained from (5.17).

5.3.2 Sub-optimal estimate

The estimate $x_e^+(k)$ in (5.8) is straightforwardly derived from the measurement output (5.2) by using the Moore-Penrose pseudoinverse C_e^+ :

$$x_e^+(k) = C_e^+ y_e(k)\,. \tag{5.21}$$

In the case where the matrix C_e of the measurement output is given by the form of a partitioned matrix $C_e = [C\ C_s]$(as (5.3)), the pseudoinverse C_e^+ can be described explicitly.

Lemma 5.1 *(Cline [22], Rao and Mitra [79])*

$$\begin{aligned}
\begin{bmatrix} C & C_s \end{bmatrix}^+ &= \begin{bmatrix} C^+ - C^+C_sU^+ \\ \qquad - C^+C_s(I-U^+U)\Gamma C_s^T(C^+)^T C^+(I-C_sU^+) \\ U^+ + (I-U^+U)\Gamma C_s^T(C^+)^T C^+(I-C_sU^+) \end{bmatrix} \\
&= \begin{bmatrix} C^+ - C^+C_sU^+ \\ \qquad - C^+C_s(I-U^+U)\Gamma C_s^T(C^+)^T C^+(I-C_sU^+) \\ C_s^+ - C_s^+C\tilde{U}^+ \\ \qquad - C_s^+C(I-\tilde{U}^+\tilde{U})\tilde{\Gamma} C^T(C_s^+)^T C_s^+(I-C\tilde{U}^+) \end{bmatrix} \\
&= \begin{bmatrix} \tilde{U}^+ + (I-\tilde{U}^+\tilde{U})\tilde{\Gamma} C^T(C_s^+)^T C_s^+(I-C\tilde{U}^+) \\ U^+ + (I-U^+U)\Gamma C_s^T(C^+)^T C^+(I-C_sU^+) \end{bmatrix},
\end{aligned} \tag{5.22}$$

where

$$\begin{aligned}
U &= (I-CC^+)C_s\,, \\
\tilde{U} &= (I-C_sC_s^+)C\,, \\
\Gamma &= [I+(I-U^+U)C_s^T(C^+)^T C^+C_s(I-U^+U)]^{-1}\,, \\
\tilde{\Gamma} &= [I+(I-\tilde{U}^+\tilde{U})C^T(C_s^+)^T C_s^+C(I-\tilde{U}^+\tilde{U})]^{-1}\,.
\end{aligned}$$

5.4 Switching Scheme of the Control Input

In the adaptive control process for the augmented control system (5.1) (=(4.3)), (5.2) (=(4.4)), the unbiased linear functional observer (4.16),(4.17) and the adaptive

state observer (4.47),(4.48) are used in order to obtain the estimate $\bar{s}(k)$ of the unknown input $s(k)$ which is the state variable of the unknown input generator (4.2). As the control input (4.54) with a pair of time-invariant gains derived by the proposed adaptive control method in Chapter 4 used this estimate $\bar{s}(k)$, the effectiveness of the control input (4.54) depends heavily on the accuracy of the estimation process. The magnitude of $e_1(k) := \bar{y}_s(k) - \hat{y}_s(k)$, which obeys the error equation (4.49), guarantees the convergence of the estimation process; therefore, the time to switch the control input from the one given by the stochastic sub-optimal regulator to the alternative given by the adaptive controller can be determined by comparing the absolute value of the estimation error e_1 with a prespecified threshold value e_{TH}.

Switching scheme of the control input

1. The augmented control system (5.1),(5.2) begins to be driven by the control input (5.8) derived from the stochastic sub-optimal regulator. At the same time, the unknown input estimation process (4.47),(4.48) with the linear functional observer (4.16),(4.17) and the parameter estimation process (4.51) begins to work.

2. The estimation error e_1 becomes smaller as time passes, and if, at k-th time step, the absolute value of the estimation error e_1 becomes smaller than a prespecified threshold value e_{TH} :

$$|e_1(k)| := |\bar{y}_s(k) - \hat{y}_s(k)| < e_{TH}\,, \tag{5.23}$$

   then, the control input is switched so that the control input (4.54) (or (4.55)) derived from the adaptive control formulation is used to drive the augmented control system from $(k+1)$-th time step.

5.5 Numerical Example

Example 5.1.

Consider the same plant as the plant (4.71) - (4.74) in Section 4.6 :

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} s(k)\,, \tag{5.24}$$

where the unknown input $s(k)$ is assumed to be the state variable of the unknown input generator of dimension 3 which is assumed to be known. Although A is unstable ($\sigma(A) = \{\pm i\}$, where i is the purely imaginary number), we can choose a feedback control gain F such that $\sigma(A+BF) = \{\pm 0.5\}$ which is a set of stable closed-loop poles because the plant (5.24) is reachable :

$$F = \begin{bmatrix} 0 & 0 \\ 1.25 & 0 \end{bmatrix}. \tag{5.25}$$

The augmented system is given by

$$x_e(k+1) = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0_{3\times 2}} & \multicolumn{3}{c}{A_s} \end{array}\right] x_e(k) + \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline \multicolumn{2}{c}{0_{3\times 2}} \end{array}\right] u(k) + \left[\begin{array}{c} 0_2 \\ \hline b_s \end{array}\right] w(k)\,, \tag{5.26}$$

$$y_e(k) = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] x_e(k)\,. \tag{5.27}$$

In (5.26), A_s , b_s are unknown parameters of an unknown input generator which is assumed to be described by 3 dimensional state space form.

(1) The control input derived from the adaptive state observer

Let

$$y_s(k) = [0\,0\,1\,0\,0]\,x_e(k)\,. \tag{5.28}$$

According to Theorem 4.2, we can obtain 2 vectors with 3 dimension $\tilde{\xi}_1, \tilde{\xi}_2$ which satisfy (4.31) as

$$\tilde{\xi}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tilde{\xi}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \tag{5.29}$$

Therefore, using Lemma 4.4, we obtain the linear functional observer as follows:

$$z(k+1) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} y_e(k) + \begin{bmatrix} 1 & 0 \end{bmatrix} u(k)\,, \tag{5.30}$$

$$\hat{y}_s(k) = z(k) + [-1\ 0\ 1]\,y_e(k)\,. \tag{5.31}$$

The unknown input generator can be written as

$$s(k+1) = \left[\begin{array}{c|cc} & 1 & 1 \\ \cline{2-3} a_s & -0.5 & 0 \\ & 0 & 0 \end{array}\right] s(k) + b_s w(k)\,, \tag{5.32}$$

where $a_s = [\,a_1\ a_2\ a_3\,]^T$ and $b_s = [\,b_1\ b_2\ b_3\,]^T$ are unknown parameters.

The estimate $\bar{s}(k)$ of the unknown input $s(k)$ is obtained by using the adaptive state observer given by

$$\bar{s}(k) = [\,\bar{y}_s(k)\ \ \bar{s}_2(k)\ \ \bar{s}_3(k)\,]^T, \tag{5.33}$$

where

$$\begin{aligned} \bar{s}_2(k) &= \bar{a}_2 v_2(k) + \bar{b}_2 t_2(k) + (-0.5)^k \bar{s}_{02}, \\ \bar{s}_3(k) &= \bar{a}_3 v_3(k) + \bar{b}_3 t_3(k), \\ \bar{y}_s(k+1) &= 0.25\bar{y}_s(k) + \{\bar{a}_1(k+1) - 0.25\}\hat{y}_s(k) \\ &\quad + \textstyle\sum_{i=2}^3 \bar{a}_i(k+1)v_i(k) + b_1(k+1)w(k) \\ &\quad + \textstyle\sum_{i=2}^3 \bar{b}_i(k+1)t_i(k) + (-0.5)^k \bar{s}_{02}, \end{aligned} \tag{5.34}$$

where $v_i\,, t_i$ $(i = 2, 3)$ are given by

$$\begin{aligned} v_2(k+1) &= -0.5v_2(k) + \hat{y}_s(k), \\ t_2(k+1) &= -0.5t_2(k) + w(k). \end{aligned} \tag{5.35}$$

$$v_3(k+1) = \hat{y}_s(k)\,,\ t_3(k+1) = w(k)\,. \tag{5.36}$$

Using $e_1(k) := \bar{y}_s(k) - \hat{y}_s(k)$, the error equation is

$$\begin{aligned} e_1(k+1) &= 0.25e_1(k) + \{\bar{a}_1(k+1) - a_1\}\hat{y}_s(k) \\ &\quad + \sum_{i=2}^{3}\{\bar{a}_i(k+1) - a_i\}v_i(k) + \{\bar{b}_1(k+1) - b_1\}w(k) \\ &\quad + \sum_{i=2}^{3}\{\bar{b}_i(k+1) - b_i\}t_i(k) + (-0.5)^k(\bar{s}_{02} - s_{02}). \end{aligned} \tag{5.37}$$

Choosing T which satisfies the conditions of Theorem 4.3 as

$$T = -E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \tag{5.38}$$

we obtain the modified control input $u_m(k)$ as

$$u_m(k) = \begin{bmatrix} 0 & 0 & 0 \\ 1.25 & 0 & 0 \end{bmatrix} y_e(k) + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \bar{s}(k). \tag{5.39}$$

(2) The control input derived from the sub-optimal regulator

Consider the following performance criterion to be minimized :

$$J(u) = E[\, x_e^T(T) Q_0 x_e(T) + \sum_{k=0}^{T-1} \{\, y_e^T(k) Q y_e(k) + u^T(k) R_u u(k) \,\}\,], \tag{5.40}$$

where

$$Q_0 = \begin{bmatrix} Q_{01} & 0 \\ 0 & 0 \end{bmatrix}, Q_{01} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 1.25 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; R_u = 0. \tag{5.41}$$

In this example, the assumptions **(A.1)**–**(A.5)** hold and $R_u \equiv 0$; therefore we can obtain the optimal gain $[\,F(k) \;\; G(k)\,]$ from (5.16) of Corollary 5.2 as

$$F(k) = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}, G(k) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, (k = 0,\, 1, \ldots,\, T-1), \tag{5.42}$$

The control input derived from the stochastic sub-optimal regulator is given by

$$u_{op}(k) = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & 0 \end{bmatrix} y_e(k). \tag{5.43}$$

(3) Simulation results

In this example, we take the controlled output $y_o(k)$ as

$$y_o(k) = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] x_e(k), \tag{5.44}$$

and consider the output dead-beat control problem. Figures 5.2 and 5.3 show the behavior of $y_{o1}(k)$ $(= x_1(k))$ and $y_{o2}(k)$ $(= x_2(k))$ of the output $y_o(k)$ of the plant (5.24) based on the proposed controller $u_m(k)$ (given by (5.39)) under unknown input, where the output $y_o(k)$ is forced to zero from the initial state $x_{e0} = [\,1\; 1\; 0\; 0\; 0\,]^T$. Figures 5.4 and 5.5 show the behavior of $y_{o1}(k)$ $(= x_1(k))$ and $y_{o2}(k)$ $(= x_2(k))$

of the output $y_o(k)$ of the plant (5.24) based on the control input $u_{op}(k)$ (given by (5.43)) derived the stochastic sub-optimal regulator with the same initial state $x_{e0} = [\,1\; 1\; 0\; 0\; 0\,]^T$. Figures 5.6 and 5.7 show the estimation processes of the unknown parameters a_s and b_s in (5.32). In this example, the true values of a_s and b_s are $a_s = [\,-1\; -0.5\; 0.1\,]^T$ and $b_s = [\,0.5\; -0.25\; 0\,]^T$. Figure 5.8 shows the transient response of the error equation (5.37), which shows the adaptive estimation error tends to zero as time steps.

On the other hand, we may point out that the estimation process includes large estimation errors until about 20 time steps from the initial time. These large estimation errors may cause the large magnitude of transient output response until about 10 time steps in Figure 5.3.

The shaping of the transient response is one of the important specifications in controller design. We use the switching scheme of the control input with u_m and u_{op} (given by (5.43)), in which u_{op} is used until $|e_1(k)| \geq e_{TH}$ and u_m is used after $|e_1(k)| < e_{TH}$. Figure 5.9 and 5.10 show the behavior of $y_{o1}(k)$ $(= x_1(k))$ and $y_{o2}(k)$ $(= x_2(k))$ of the output $y_o(k)$ of the plant (5.24) based on the switching scheme of the control input with u_{op} and u_m under unknown input, taking $e_{TH} = 0.2$.

5.6 Conclusion

We have considered the control problem of uncertain linear systems described by difference equations which contain unknown parameters. For these systems, the dual controller has been proposed which implements efficient control actions by using two kinds of controllers – the stochastic sub-optimal regulator and the deterministic adaptive regulator – and switching them effectively.

It is well-known that the separation principle holds for the ordinary stochastic optimal regulator problem. But for the problem formulation of this chapter, this principle does not hold because the optimal estimate of the state of the augmented system cannot be constructed by using an ordinary state observer. It is one of open problems to develop a scheme of the optimal state estimation for linear systems with unknown system parameters.

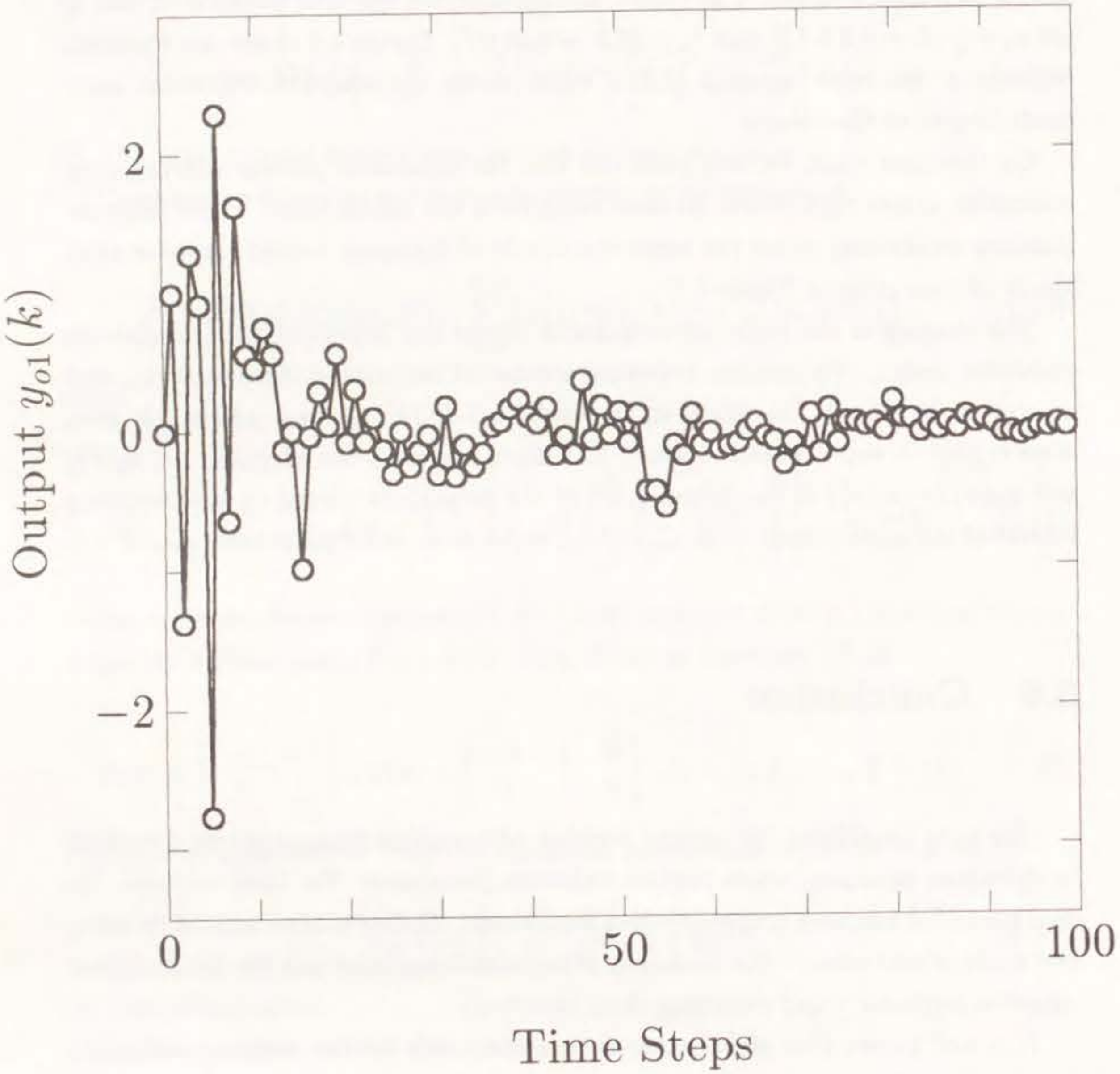

Figure 5.2: Behavior of the output $y_{o1}(k)$ based on the adaptive controller $u_m(k)$(Example 5.1).

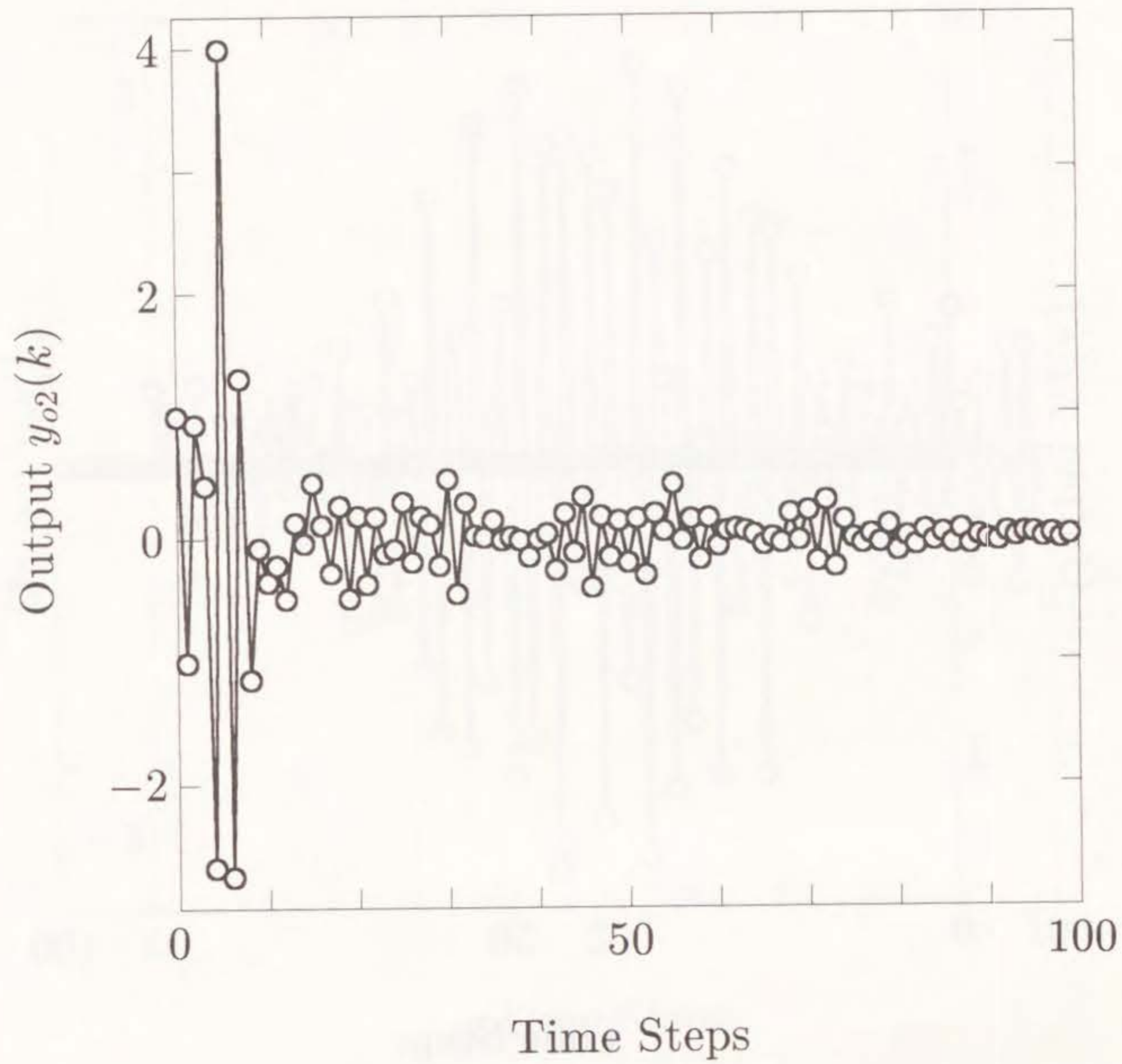

Figure 5.3: Behavior of the output $y_{o2}(k)$ based on the adaptive controller $u_m(k)$(Example 5.1).

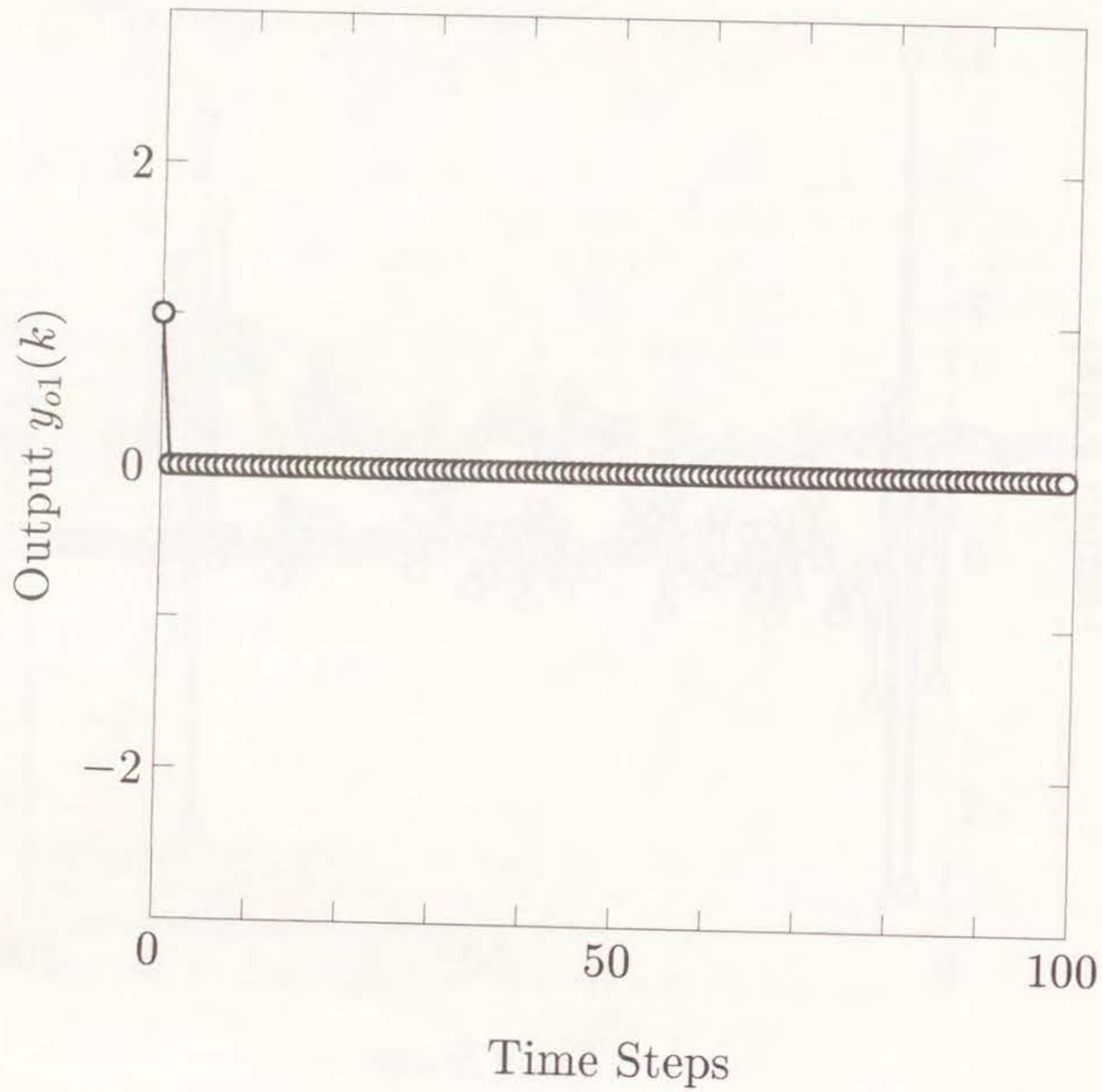

Figure 5.4: Behavior of the output $y_{o1}(k)$ based on the stochastic sub-optimal controller $u_{op}(k)$(Example 5.1).

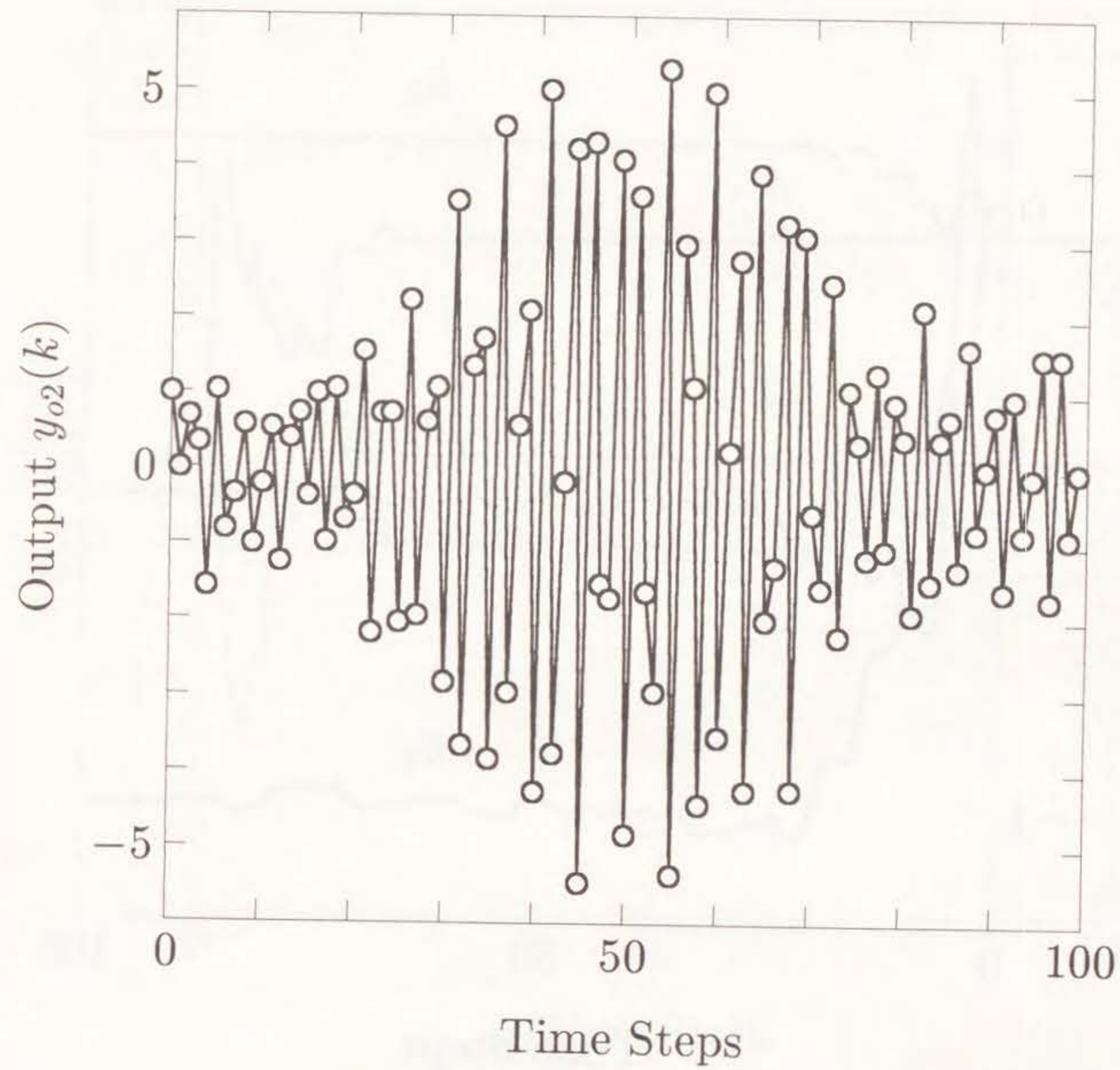

Figure 5.5: Behavior of the output $y_{o2}(k)$ based on the stochastic sub-optimal controller $u_{op}(k)$(Example 5.1).

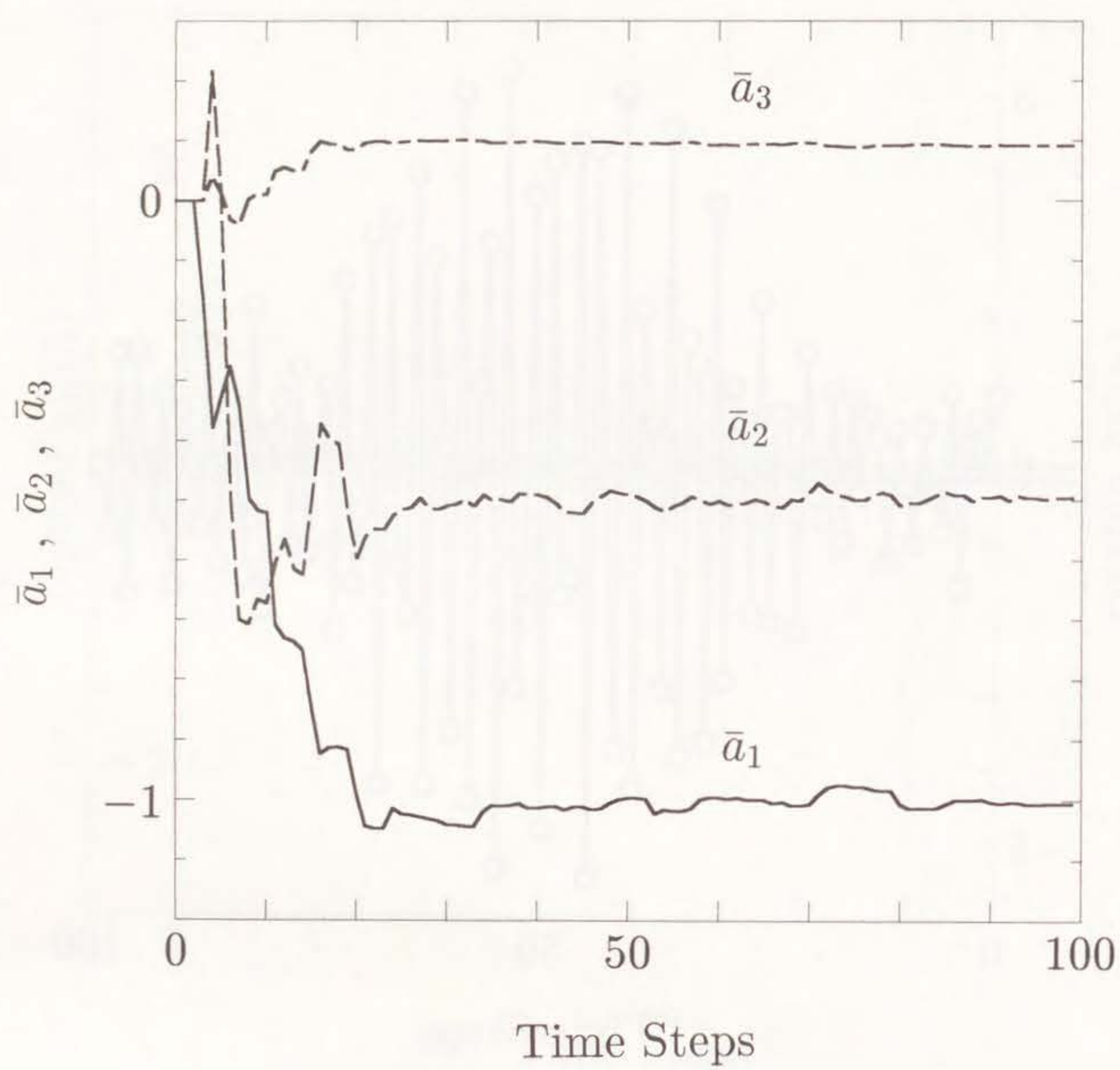

Figure 5.6: Estimation process of the parameter a_s(Example 5.1).

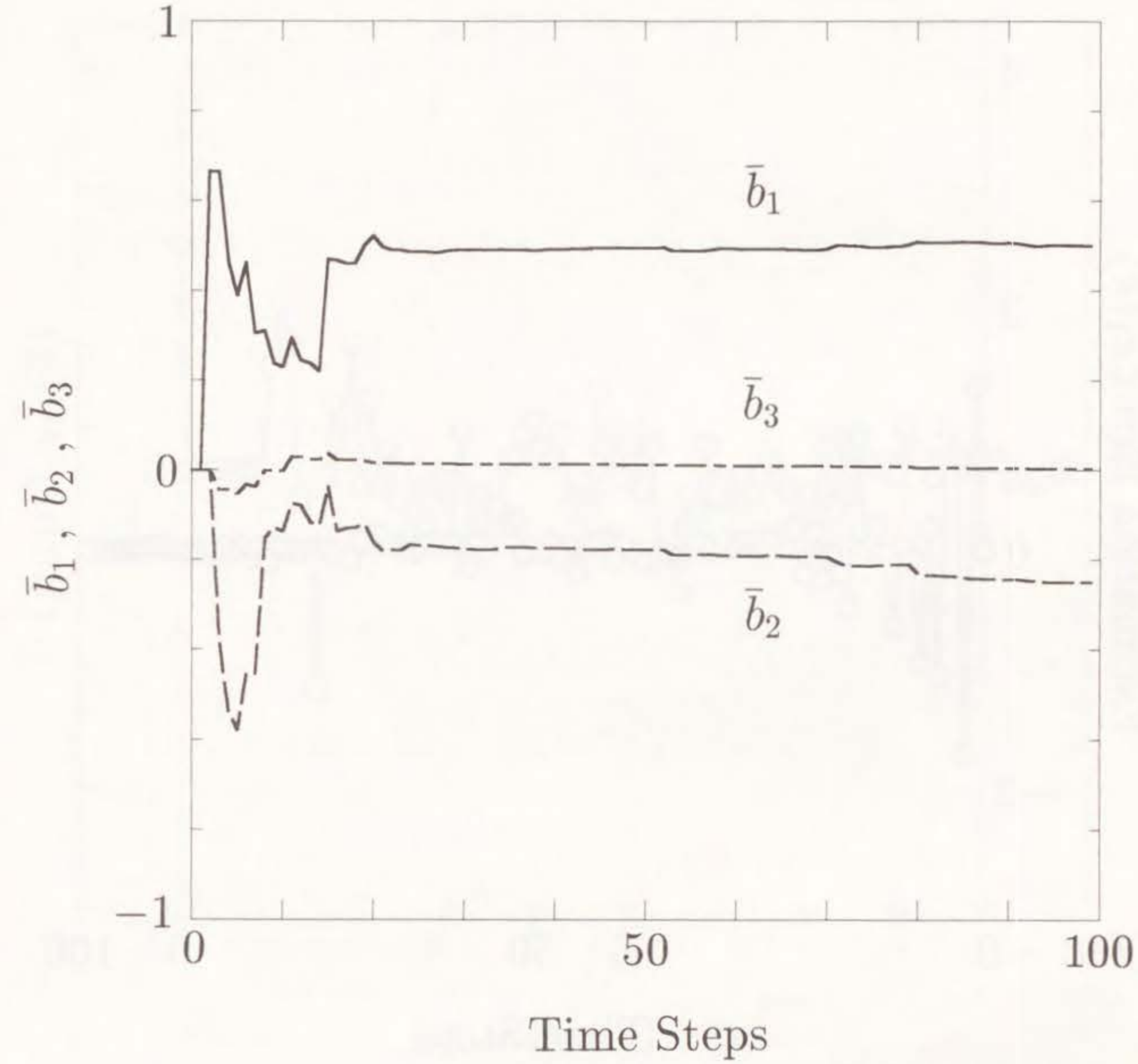

Figure 5.7: Estimation process of the parameter b_s(Example 5.1).

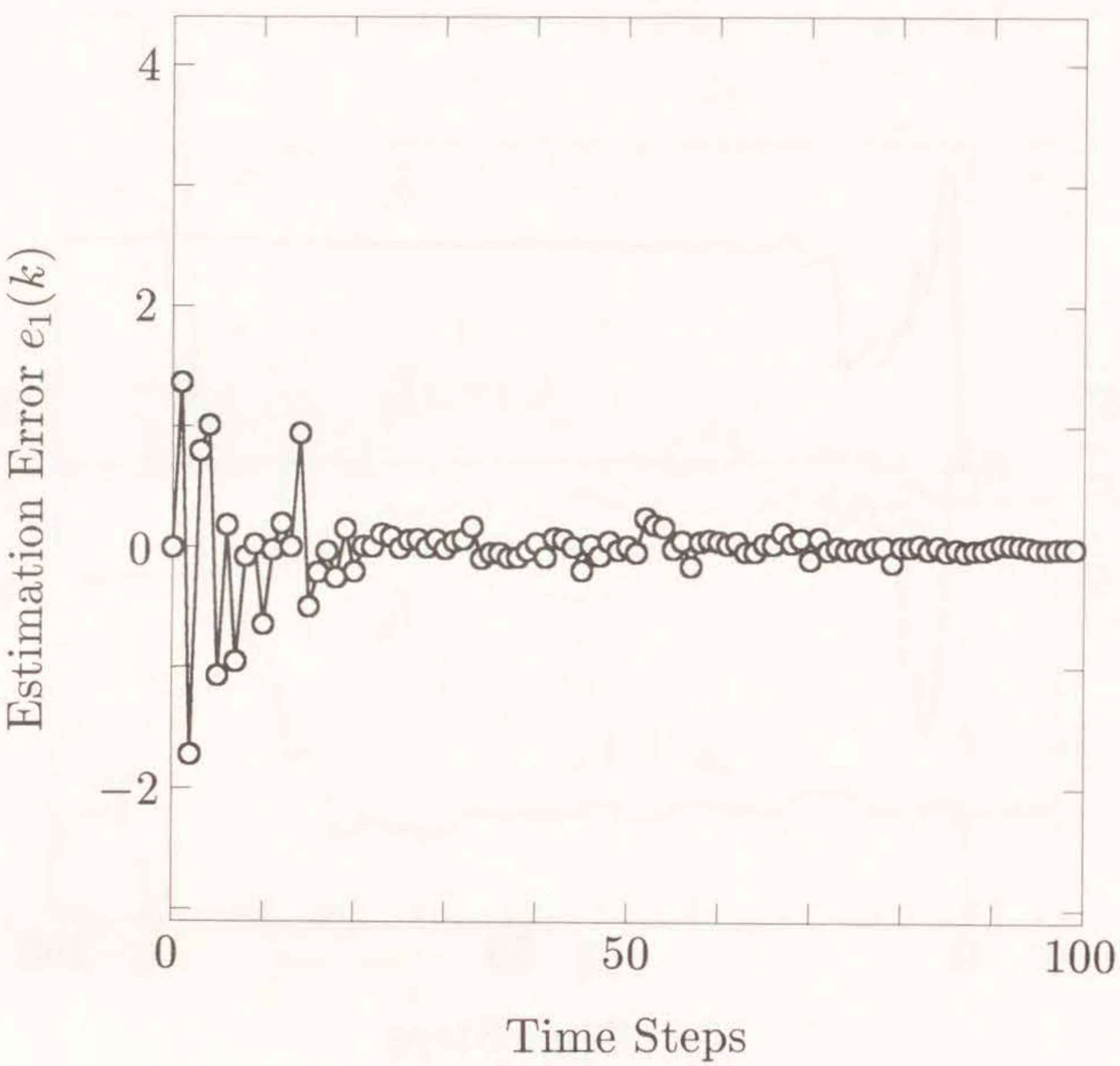

Figure 5.8: Behavior of the adaptive estimation error $e_1(k)$(Example 5.1).

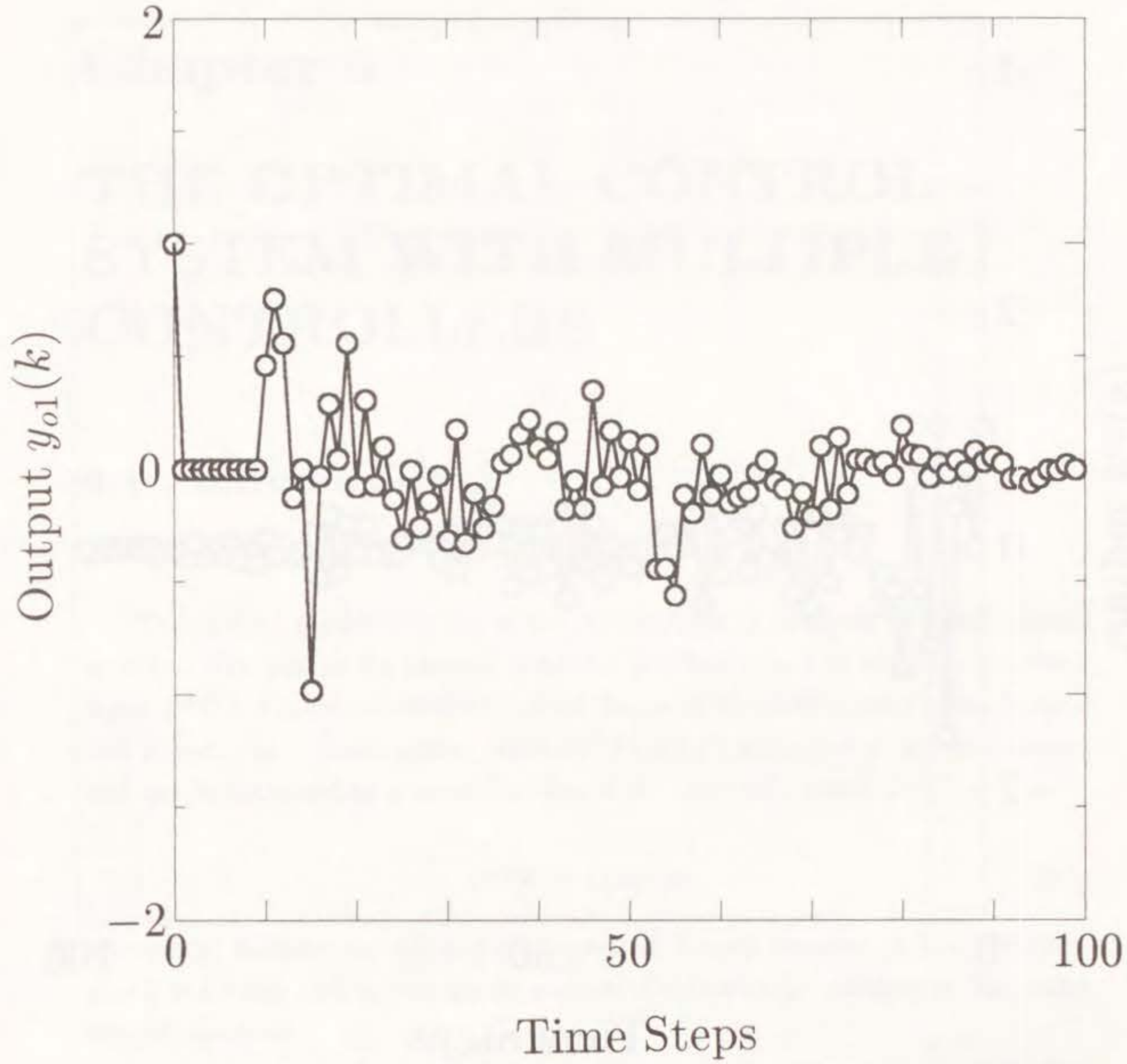

Figure 5.9: Behavior of the second element $y_{o1}(k)$ of the output $y_o(k)$ based on the dual controller $u_{op}(k)$ and $u_m(k)$(Example 5.1).

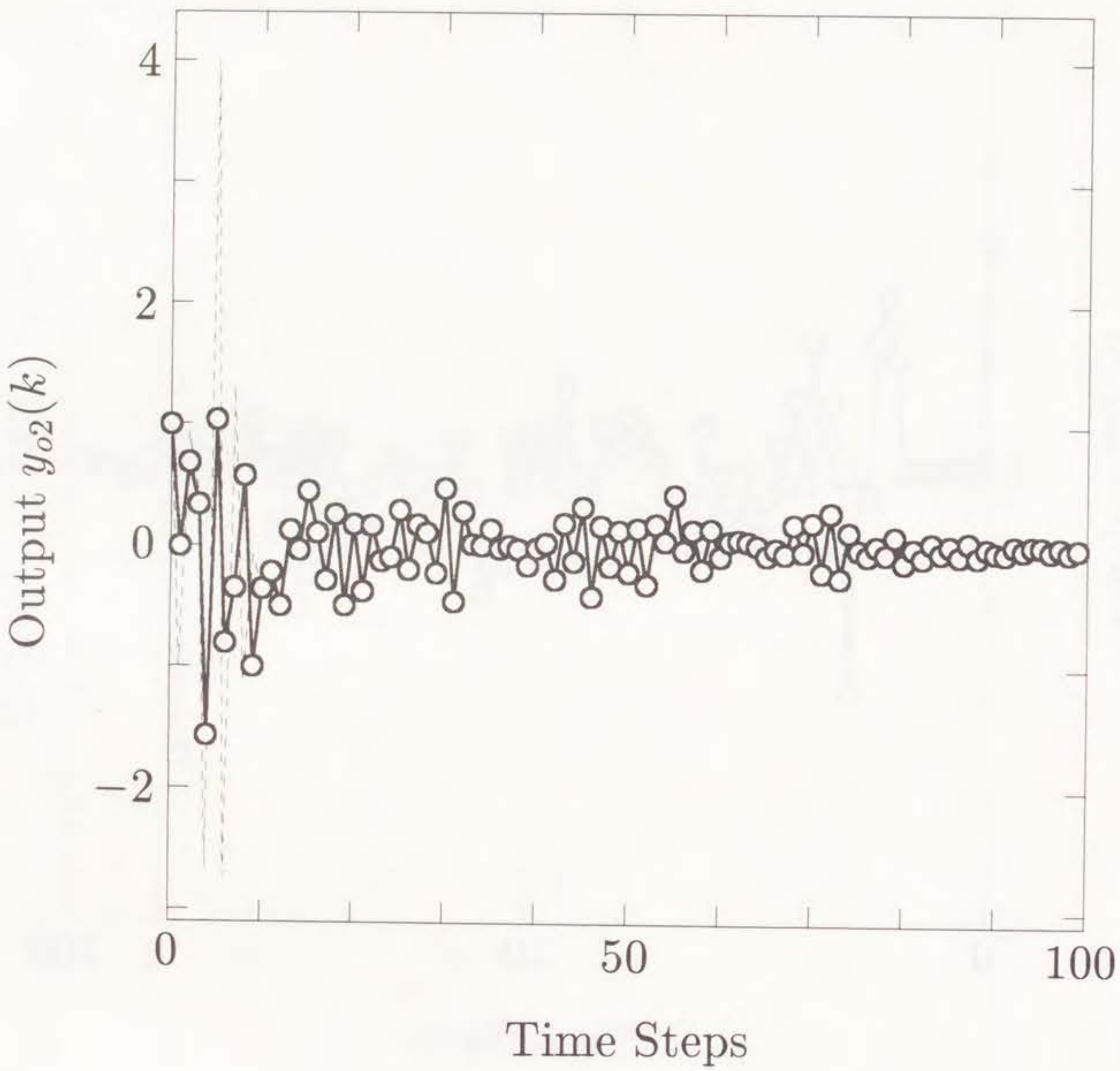

Figure 5.10: Behavior of the second element $y_{o2}(k)$ of the output $y_o(k)$ based on the dual controller $u_{op}(k)$ and $u_m(k)$(Example 5.1).

Chapter 6

THE OPTIMAL CONTROL SYSTEM WITH MULTIPLE CONTROLLERS

6.1 Introduction

The optimal regulator is one of the powerful design methods for linear control systems. The goal of the optimal regulator problem is to find an optimal control input $u^{opt}(k)$, if it exists, which minimizes the quadratic performance index J. As is well known, the optimal control input $u^{opt}(k)$, which minimizes J, uniquely exists, and can be expressed as a linear function of the state $x(k)$ (see (1.29)):

$$u^{opt}(k) = L(k)x(k) \,, \tag{6.1}$$

where $L(k)$ includes the solution to the matrix Riccati equation (1.31). We know that it is a tough task to execute the process of obtaining the solution to the matrix Riccati equation.

As the dimension of a dynamic system to be controlled becomes large, it is often too costly (sometimes practically impossible) to have only one decision maker (or controller) in the system who possesses all available information on the system and make all the decisions for the system, for example, the optimal control input. Hence, much literature has been devoted to control problems of the system with multiple control agents or decentralized dynamic systems.

In this chapter, we consider the optimization problem with quadratic criterion for linear multivariable control systems with multiple control agents. The structure of the optimal control system with multiple controllers is derived, which is globally equivalent to the optimal control system with single controller. Figure 6.1 illustrates the optimal control system with multiple control agents.

The organization of this chapter is as follows. In Section 6.2, we describe some preliminary results on the optimal regulator with single controller and the state space orthogonal decomposition with respect to the real symmetric matrix which is used as the weighting matrix in the performance index to be minimized. The definition of the optimal control problem for subsystems and the problem formulation of the control optimal decoupling is given in Section 6.3. In Section 6.4, we consider the solvability condition of the control optimal decoupling problem. A numerical example is given in Section 6.5.

6.2 Optimal Regulator and Preliminaries

Consider the linear system :

$$x(k+1) = Ax(k) + Bu(k)\ ,\quad x(0) = x_0\ , \tag{6.2}$$

with the performance index to be minimized :

$$J(u) = \sum_{k=0}^{\infty}[\, x^T(k)Qx(k) + u^T(k)Ru(k)\,]\ , \tag{6.3}$$

where $x \in \mathbf{R}^n$ is the state of the system to be controlled, $u \in \mathbf{R}^r$ the control input, and $Q \in \mathbf{R}^{n\times n}$ are positive semi-definite symmetric matrix, $R \in \mathbf{R}^{r\times r}$ is positive definite symmetric matrix.

It is assumed that the pair $(A,\, B)$ is reachable and, for any matrix D such that $D^TD = Q$, the pair $(D^T,\, A)$ is observable. The latter assumption is sufficient to guarantee the asymptotic stability of the closed-loop system. For this optimal control problem, the following proposition is known(Anderson and Moore [7]).

Proposition 6.1 *The optimal control input $u^{opt}(k)$ with respect to (6.2)-(6.3) is given by the state feedback control*

$$u^{opt}(k) = K^{opt}x(k)\ , \tag{6.4}$$

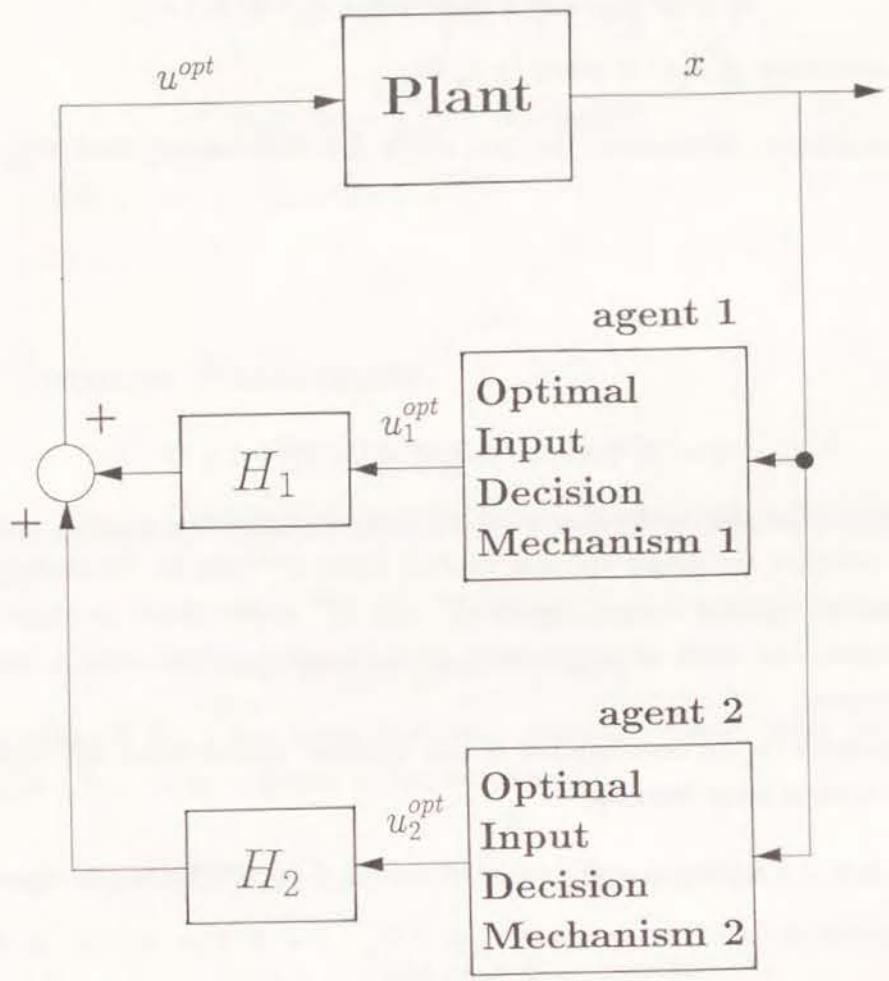

Figure 6.1: Block diagram of the control system with multiple control agents.

where K^{opt} is the optimal time-invariant feedback control gain given by

$$K^{opt} = -(B^T \Phi B + R)^+ B^T \Phi A \,, \tag{6.5}$$

in which Φ is the unique real symmetric solution to the discrete matrix Riccati equation :

$$\Phi = A^T \Phi A - A^T \Phi B (B^T \Phi B + R)^+ B^T \Phi A + Q \tag{6.6}$$

and the minimum of $J(u)$ is given by $x_0^T \Phi x_0$.

In this chapter, we consider the case where R is null matrix, thus (6.3) and (6.6) will be

$$J_x = \sum_{k=0}^{\infty} x^T(k) Q x(k) \,, \tag{6.7}$$

and

$$\Phi = A^T \Phi A - A^T \Phi B (B^T \Phi B)^+ B^T \Phi A + Q \,, \tag{6.8}$$

respectively. For this optimal control problem formulation, we will consider the problem whether the single optimal control input u^{opt} can be decoupled into two noninteractive optimal control inputs u_1^{opt} and u_2^{opt} where these decoupled optimal control inputs can work as equivalently as the single optimal control input in the control process.

Preparatory to the investigation of this optimal control input decoupling problem, it is needed some lemmas.

Lemma 6.1 *An arbitrary real symmetric matrix L ($\in \mathbf{R}^{n \times n}$) can be represented by*

$$L = \sum_{i=1}^{n} \lambda_i t_i t_i^T \,, \tag{6.9}$$

where λ_i ($i \in \underline{n}$) are the real eigenvalues of L and t_i ($\in \mathbf{R}^n, i \in \underline{n}$) are the eigenvectors of L corresponding to λ_i.

Lemma 6.1 is known as the spectral resolution of the real symmetric matrix [90].

Lemma 6.2 *The matrix $t_i t_i^T$ is the projection matrix on $Im(t_i)$, that is ,*

$$t_i t_i^T = P_{Im(t_i)} =: P_{(t_i)} \,.$$

Proof. From the decomposition of projection matrix, $P_{(t_i)}$ can be written as

$$P_{(t_i)} = t_i (t_i^T t_i)^{-1} t_i^T \,.$$

Since $t_i^T t_i = 1$, $P_{(t_i)} = t_i t_i^T$.
Applying Lemmas 6.1 and 6.2 to the quadratic term $x^T(k) Q x(k)$ in (6.7), it follows that

$$x^T(k) Q x(k) = \sum_{i=1}^{n} \gamma_i x^T(k) P_{(t_i)} x(k) \,, \tag{6.10}$$

where γ_i ($i = 1, 2, \ldots, n$) are eigenvalues of Q.

6.3 Problem Statement

Now we introduce the following assumption. Given any subspaces $\mathcal{F}_1$ and $\mathcal{F}_2$ ($\subset \mathcal{X}$) such that

$$\mathcal{X} = \mathcal{F}_1 \bigoplus \mathcal{F}_2 \text{ and } \mathcal{F}_1^{\perp} = \mathcal{F}_2 \,. \tag{6.11}$$

This decoupling of the state space is usually possible, indeed, since for $i \neq j$, $\mathrm{Im}\{P_{(i)}\} \perp \mathrm{Im}\{P_{(j)}\}$, it is sufficient to set, for example,

$$\mathcal{F}_1 = \mathrm{Im}\{P_{(t_1, t_2, \cdots, t_\pi)}\} \text{ and } \mathcal{F}_2 = \mathrm{Im}\{P_{(t_{\pi+1}, t_{\pi+2}, \cdots, t_n)}\} \,, \tag{6.12}$$

where π is an integer such that $0 < \pi < n$. According to the given subspaces $\mathcal{F}_1$ and $\mathcal{F}_2$, (6.10) can be decoupled into two terms :

$$\begin{aligned} x^T(k) Q x(k) &= \textstyle\sum_{i=1}^{\pi} \gamma_i x^T(k) P_{(t_i)} x(k) + \sum_{i=\pi+1}^{n} \gamma_i x^T(k) P_{(t_i)} x(k) \\ &=: x^T(k) \Lambda_1 x(k) + x^T(k) \Lambda_2 x(k) \end{aligned} \tag{6.13}$$

such that $x^T(\cdot) \Lambda_j x(\cdot)$ depends only on $\mathcal{F}_j$ ($j = 1, 2$).

Based on this decoupling of the state space, for some linear maps $H_j : \mathcal{U} \to \mathcal{U}$ ($j = 1, 2$), the following two optimal control subsystems are considered with the performance index to be minimized.

Subsystem j $(j=1,2)$:

$$x(k+1) = Ax(k) + BH_j u_j(k), \quad x(0) = x_{j0} = P_{\mathcal{F}_j} x_0 \in \mathcal{F}_j \tag{6.14}$$

Performance index to be minimized for the subsystem j $(j=1,2)$:

$$J_{jx} = \sum_{k=0}^{\infty} \{x^T(k)\Lambda_j x(k)\}. \tag{6.15}$$

Now we assume that, for $j = 1, 2$,

1. The pair (A, BH_j) is reachable in $\mathcal{F}_j$.

2. The pair $(\Lambda_j^{\frac{1}{2}}, A)$ is observable in $\mathcal{F}_j$.

Then, discrete Riccati equation for the subsystem j $(j=1,2)$ is given by

$$\Phi_j = A^T\Phi_j A - A^T\Phi_j BH_j[(BH_j)^T\Phi_j BH_j]^{+}(BH_j)^T\Phi_j A + \Lambda_j, \tag{6.16}$$

The problem which we are of interest is as follows.

[**Control Optimal Decoupling Problem (CODP)**]

Given a linear discrete-time system (6.2), find (if possible) the linear map H_j : $\mathcal{U} \to \mathcal{U}$ $(j=1,2)$ such that the optimal input u^{opt} with respect to (6.2) and (6.7) can be replaced by the linear combination $H_1 u_1^{opt} + H_2 u_2^{opt}$, where u_j^{opt} $(j=1,2)$ are the optimal control inputs with respect to (6.14) and (6.15).

6.4 Solvability Condition of CODP

In order to solve the problem of control optimal decoupling, the supremal element of the class of A-*invariant* subspaces is introduced to consider the optimal control input in a decoupled subspace of the state space. For the subclass of A-*invariant* subspaces contained in a subspace $\mathcal{G}\,(\subset \mathcal{X})$, denoted by $\underline{\mathcal{I}}(A;\mathcal{G})$, Propositions 1.1 and 1.2 guarantee the existence of the supremal element

$$\mathcal{V}^* := sup\,\underline{\mathcal{I}}(A;\mathcal{G}). \tag{6.17}$$

For the computation of $\mathcal{V}^*$, we have the following lemma.

Lemma 6.3 *Let $A : \mathcal{X} \to \mathcal{X}$, and $\mathcal{G} \subset \mathcal{X}$. Define the sequence $\mathcal{V}^\mu$ according to*

$$\begin{aligned} \mathcal{V}^0 &= \mathcal{G}, \\ \mathcal{V}^\mu &= \mathcal{G} \cap A^{-1}\mathcal{V}^{\mu-1}, \\ &\quad (\mu = 1, 2, \ldots, n). \end{aligned} \tag{6.18}$$

Then $\mathcal{V}^\mu \subset \mathcal{V}^{\mu-1}$, and for some $k \le dim(\mathcal{G})$,

$$\mathcal{V}^k = sup\,\underline{\mathcal{I}}(A;\mathcal{G}). \tag{6.19}$$

Proof. We first observe that the sequence $\mathcal{V}^\mu$ is nonincreasing. Clearly $\mathcal{V}^1 \subset \mathcal{V}^0$, and if $\mathcal{V}^\mu \subset \mathcal{V}^{\mu-1}$, then

$$\mathcal{V}^{\mu+1} = \mathcal{G} \cap A^{-1}\mathcal{V}^\mu \subset \mathcal{G} \cap A^{-1}\mathcal{V}^{\mu-1} = \mathcal{V}^\mu.$$

Thus for some $k \le dim(\mathcal{G})$, $\mathcal{V}^\mu = \mathcal{V}^k$ $(\mu \ge k)$. Now $\mathcal{V} \in \underline{\mathcal{I}}(A;\mathcal{G})$ if and only if

$$\mathcal{V} \subset \mathcal{G} \ \text{ and } \ \mathcal{V} \subset A^{-1}\mathcal{V}. \tag{6.20}$$

Therefore,

$$\mathcal{V} \subset \mathcal{V}^k \in \underline{\mathcal{I}}(A;\mathcal{G}),$$

and as $\mathcal{V}$ is arbitrary, the result follows.

We also obtain the following theorem.

Theorem 6.1 *If there exists $\mathcal{R}$ such that for an arbitrary $\mathcal{G} \subset \mathcal{X}$,*

$$\langle A \,|\, \mathcal{B} \cap \mathcal{R}\rangle \subset \mathcal{G},$$

there exists a linear map $H : \mathcal{U} \to \mathcal{U}$ such that

$$\langle A \,|\, Im(BH)\rangle = \langle A \,|\, \mathcal{B} \cap \mathcal{R}\rangle. \tag{6.21}$$

Proof. Let $\{b_1, b_2, \cdots, b_m\}$ be a basis for $\mathcal{B} \cap \mathcal{R}$. Then $b_i = Bu_i$ $(u_i \in \mathcal{U})$, where these u_i $(i \in \underline{m})$ are independent. Let $\{u_1, \cdots, u_m, \cdots, u_r\}$ be a basis for $\mathcal{U}$, and define H such that

$$\begin{aligned} Hu_i &= u_i \quad (i = 1, 2, \ldots, m), \\ Hu_i &= 0 \quad (i = m+1, m+2, \ldots, r). \end{aligned}$$

Then we have

$$\operatorname{Im}(BH) = \mathcal{B} \cap \mathcal{R}.$$

When we consider the problem of control input optimal decoupling, the following lemma is useful.

Lemma 6.4 *Let $\mathcal{F}_j$ $(j = 1,2)$ be any subspaces which satisfy (6.11). For any real symmetric matrix Φ and the matrix S such that $\Phi = S^T S$, there exist the real symmetric matrices Φ_j $(j = 1,2)$ and the matrices S_j $(j = 1,2)$ so that the following relations hold :*

$$\begin{array}{rcll} \Phi & = & \Phi_1 + \Phi_2 , & \\ \Phi_j & = & S_j^T S_j & (j = 1,2), \\ S_i^T S_j & = & 0 & (i,j = 1,2,\ i \neq j), \\ Im\,\Phi_j & \subset & \mathcal{F}_j , & (j = 1,2). \end{array} \tag{6.22}$$

Proof. The matrix S_j can be constructed by using the projection matrix in Lemma 6.2. The relations of (6.22) are proved straightforward from the property of the decoupled spaces $\mathcal{F}_j$ $(j = 1,2)$.

In Theorem 6.1, setting $\mathcal{G} = \mathcal{F}_j$ $(j = 1,2)$ and

$$\mathcal{R} = \mathcal{V}_j^* = sup\, \underline{\mathcal{I}}(A\,;\mathcal{F}_j)\,, \quad (j = 1,2)\,,$$

defined by (6.17) and noting that Lemma 6.4, we obtain the following theorem as the solution of the control optimal decoupling problem.

Theorem 6.2 *Let Φ, Φ_j, $(j = 1,2)$ be the real symmetric (time-invariant) solutions of the discrete Riccati equations with respect to the overall system (6.2)- (6.7) and the subsystems (6.14), respectively. Then, there exist the maps H_j $(j = 1,2)$ that give the solution of the control optimal decoupling problem (CODP) if the linear maps H_j $(j = 1,2)$ are chosen so that*

$$\langle A \,|\, Im(BH_j)\rangle = \langle A \,|\, \mathcal{B} \cap \mathcal{V}_j^*\rangle = \mathcal{F}_j\,, \quad (j = 1,2)\,, \tag{6.23}$$

where

$$\mathcal{V}_j^* = sup\, \underline{\mathcal{I}}(A\,;\mathcal{F}_j)\,, \quad (j = 1,2)\,.$$

Proof. Substituting the matrices S and S_j $(j = 1,2)$ into Φ and Φ_j $(j = 1,2)$ of the corresponding Riccati equations (6.8) and (6.16) yields

$$S^T S = A^T S^T S A - A^T S^T S B (B^T S^T S B)^+ B^T S^T S A + Q\,, \tag{6.24}$$

$$\begin{array}{rcl} S_j^T S_j & = & A^T S_j^T S_j A - A^T S_j^T S_j B H_j \\ & & \times\, (H_j^T B^T S_j^T S_j B H_j)^+ H_j^T B^T S_j^T S_j A \\ & & +\, \Lambda_j\,, \quad (j = 1,2)\,. \end{array} \tag{6.25}$$

CODP is solvable if

1. (6.24) is equivalent to

$$\begin{array}{rcl} \sum_{j=1}^{2} S_j^T S_j & = & \sum_{j=1}^{2} \{ A^T S_j^T S_j A - A^T S_j^T S_j B H_j \\ & & \times\, (H_j^T B^T S_j^T S_j B H_j)^+ H_j^T B^T S_j^T S_j A \\ & & +\, \Lambda_j \}\,. \end{array} \tag{6.26}$$

2. $H_1 u_1^{opt} + H_2 u_2^{opt}$, which is the linear combination of the optimal control inputs (u_j^{opt} $(j = 1,2)$) with respect to (6.14) and (6.15) is the optimal control input for the linear system (6.2) with the performance index (6.7).

Obviously, we see from (6.22) of Lemma 6.4 that

$$\Phi = S^T S = (S_1 + S_2)^T (S_1 + S_2) = S_1^T S_1 + S_2^T S_2 = \Phi_1 + \Phi_2\,. \tag{6.27}$$

For the map H_j $(j = 1,2)$, it is easily shown that the following relations hold.

$$\begin{array}{rcll} \operatorname{Im}(BH_j) & \subset & \mathcal{F}_j\,, & (j = 1,\,2)\,, \\ \operatorname{Im}(\Phi_j B) & = & \operatorname{Im}(\Phi_j B H_j)\,, & (j = 1,\,2)\,, \\ \Phi_i B H_j & = & 0\,, & (i,\,j = 1,\,2\; i \neq j)\,. \end{array}$$

Then, from some computation, if the equality

$$S^T S B (B^T S^T S B)^+ B^T S^T S = \sum_{j=1}^{2} \{ S_j^T S_j B H_j (H_j^T B^T S_j^T S_j B H_j)^+ H_j^T B^T S_j^T S_j \}\,, \tag{6.28}$$

holds, we can conclude that (6.24) is equivalent to (6.26).

Noting that the relations between the orthogonal projector and the Moore-Penrose pseudoinverse (see (1.9) and (1.10)), it follows that

$$SB(B^TS^TSB)^+B^TS^T = P_{\mathrm{Im}SB}\,, \tag{6.29}$$

and for $j = 1, 2$,

$$S_jBH_j(H_j^TB^TS_j^TS_jBH_j)^+H_j^TB^TS_j^T = P_{\mathrm{Im}S_jBH_j}\,, \tag{6.30}$$

$$P_{\mathrm{Im}S_jBH_j} = P_{\mathrm{Im}SBH_j} = P_{\mathrm{Im}S_jB}\,, \tag{6.31}$$

$$P_{\mathrm{Im}S_1BH_1} + P_{\mathrm{Im}S_2BH_2} = P_{\mathrm{Im}SB}\,, \tag{6.32}$$

$$S_i^TP_{\mathrm{Im}S_jBH_j}S_i = 0\,,\ (i \neq j)\,, \tag{6.33}$$

$$S_i^TP_{\mathrm{Im}S_jBH_j}S_j = 0\,,\ (i \neq j)\,. \tag{6.34}$$

From (6.30), (6.33) and (6.34), we obtain

$$\begin{aligned}\text{The right-hand side of (6.28)} &= \textstyle\sum_{j=1}^2\{S_j^TP_{\mathrm{Im}S_jBH_j}S_j\}\\ &= (S_1+S_2)^T\{P_{\mathrm{Im}S_1BH_1} + P_{\mathrm{Im}S_2BH_2}\}(S_1+S_2)\,.\end{aligned}$$

Noting that (6.32), (6.33) and (6.34), it follows that

$$\begin{aligned}&(S_1+S_2)^T\{P_{\mathrm{Im}S_1BH_1} + P_{\mathrm{Im}S_2BH_2}\}(S_1+S_2)\\ &\qquad = S^TP_{\mathrm{Im}SB}S\\ &\qquad = \text{the left-hand side of (6.28)}\,.\end{aligned}$$

Thus, we have shown that (6.26) is equivalent to (6.24).

Taking account of the form of optimal regulator given by Proposition 6.1, it turns out that

$$H_1u_1^{opt}(k) + H_2u_2^{opt}(k) = -\sum_{j=1}^2 H_j(H_j^TB^TS_j^TS_jBH_j)^+H_j^TB^TS_j^TS_jAx(k)\,. \tag{6.35}$$

Using the following relations [13],

$$(H_j^TB^TS_j^TS_jBH_j)^+H_j^TB^TS_j^T = (S_jBH_j)^+\,,$$

$$(S_jBH_j)^+S_i = 0\,,\ H_i(S_jBH_j)^+ = 0\,,\ (i \neq j)\,,$$

$$(SBH_1)^+ + (SBH_2)^+ = \{SB(H_1+H_2)\}^+$$

and noticing that the equality $(HH^T(HH^T)^+ = I_r$ where $H := H_1 + H_2$ is nonsingular and

$$H^T(HH^T)^+\{(SB)^TSB\}^+(SB)^T = (SBH)^+\,,$$

it follows that

$$\begin{aligned}&\sum_{j=1}^2 H_j(H_j^TB^TS_j^TS_jBH_j)^+H_j^TB^TS_j^TS_jA\\ &= H_1(S_1BH_1)^+S_1A + H_2(S_2BH_2)^+S_2A\\ &= (H_1+H_2)(S_1BH_1)^+(S_1+S_2)A + (H_1+H_2)(S_2BH_2)^+(S_1+S_2)A\\ &= (H_1+H_2)\{(S_1BH_1)^+ + (S_2BH_2)^+\}(S_1+S_2)A\\ &= (H_1+H_2)(\{(S_1+S_2)BH_1\}^+ + \{(S_1+S_2)BH_2\}^+)(S_1+S_2)A\\ &= (H_1+H_2)\{(SBH_1)^+ + (SBH_2)^+\}SA\\ &= (H_1+H_2)\{SB(H_1+H_2)\}^+SA\\ &= (H_1+H_2)(H_1+H_2)^T\{(H_1+H_2)(H_1+H_2)^T\}^+\\ &\qquad \times\{(SB)^TSB\}^+(SB)^TSA\\ &= \{(SB)^TSB\}^+(SB)^TSA\,,\end{aligned} \tag{6.36}$$

which shows that $H_1u_1^{opt} + H_2u_2^{opt}$ is the optimal control input for the linear system (6.2) with the performance index (6.7).

6.5 Numerical Example

Example 6.1.

Consider the linear system

$$x(k+1) = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}x(k) + \begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}u(k)\,, \tag{6.37}$$

with the performance index to be minimized :

$$J_x = \sum_{k=0}^{\infty} x^T(k) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k). \tag{6.38}$$

Using Proposition 6.1, the solution Φ to Riccati equation with respect to the system (6.37)-(6.38) is given by

$$\Phi = \begin{bmatrix} 2.618 & 0 & 1.618 \\ 0 & 2 & 0 \\ 1.618 & 0 & 2.618 \end{bmatrix}. \tag{6.39}$$

Since the eigenvalues of the weighting matrix Q in (6.38) are $\{2, 1, 0\}$, by Lemma 6.1, the matrix Q can be decoupled as follows :

$$\begin{aligned} Q &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{6.40}$$

The transformation matrix T such that

$$\mathrm{diag}\,\{1, 2, 1\} = T^{-1} Q T,$$

is given by the unit matrix I_3.

Let the subspace $\mathcal{F}_1$ and $\mathcal{F}_2$ be the decoupled subspace of the state space $\mathcal{X}$ which satisfy (6.11) :

$$\mathcal{F}_1 = \mathrm{Im}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\right), \quad \mathcal{F}_2 = \mathrm{Im}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right). \tag{6.41}$$

Now consider the following two optimal control subsystems.

$$x(k+1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} H_j u_j(k), \quad (j = 1, 2), \tag{6.42}$$

with the performance index to be minimized :

$$J_{jx} = \sum_{k=0}^{\infty} x^T(k) \Lambda_j x(k), \quad (j = 1, 2), \tag{6.43}$$

where

$$\Lambda_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.44}$$

Using $\mathcal{V}_j^* = \mathcal{F}_j, (j = 1, 2)$ in Theorem 6.2, we can obtain the maps H_j $(j = 1, 2)$ as

$$H_1 = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}, \; H_2 = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix}, \quad (\alpha \neq 0, \beta \neq 0). \tag{6.45}$$

The Riccati equations of the optimal control problem for the subsystems (6.42)-(6.44) with (6.45) have the solutions Φ_j $(j = 1, 2)$ as

$$\begin{aligned} \Phi_1 &= \begin{bmatrix} 2.618 & 0 & 1.618 \\ 0 & 0 & 0 \\ 1.618 & 0 & 2.618 \end{bmatrix} \\ &= \begin{bmatrix} 1.618 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1.272 \end{bmatrix} \begin{bmatrix} 1.618 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1.272 \end{bmatrix} = S_1^T S_1, \end{aligned} \tag{6.46}$$

$$\Phi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = S_2^T S_2. \tag{6.47}$$

As is shown, the matrices Φ_j $(j = 1, 2)$ have the decomposition matrices S_j $(j = 1, 2)$ satisfying Lemma 6.4. And the matrix Φ has the decomposition matrix S which is $S_1 + S_2$:

$$\begin{aligned} \Phi = S^T S &= \begin{bmatrix} 1.618 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1.272 \end{bmatrix} \begin{bmatrix} 1.618 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1.272 \end{bmatrix} \\ &= (S_1 + S_2)^T (S_1 + S_2). \end{aligned} \tag{6.48}$$

Let $u^{opt}(k), u_j^{opt}(k)$ $(j = 1, 2)$ be the optimal control input for the system (6.37)-(6.38) and the subsystem j $(j = 1, 2)$ (6.42)-(6.43), respectively. Then we obtain that

$$\begin{aligned} u^{opt}(k) &= -(B^T \Phi B)^+ B^T \Phi A x(k), \\ &= \begin{bmatrix} -0.618 & 0 & -1.618 \\ 0 & -1 & 0 \end{bmatrix} x(k), \end{aligned} \tag{6.49}$$

$$\begin{aligned} u_1^{opt}(k) &= -[(BH_1)^T \Phi_1 B H_1]^+ (BH_1)^T \Phi_1 A x(k), \\ &= \frac{1}{\alpha} \begin{bmatrix} -0.618 & 0 & -1.618 \\ 0 & 0 & 0 \end{bmatrix} x(k), \quad (\alpha \neq 0), \end{aligned} \tag{6.50}$$

$$
\begin{aligned}
u_2^{opt}(k) &= -[(BH_2)^T\Phi_2 BH_2]^+(BH_2)^T\Phi_2 Ax(k), \\
&= \frac{1}{\beta}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} x(k), \quad (\beta \neq 0).
\end{aligned} \tag{6.51}
$$

Finally, we can easily check that the equation

$$u^{opt}(k) = H_1 u_1^{opt}(k) + H_2 u_2^{opt}(k) \tag{6.52}$$

is satisfied by using (6.45) and (6.49)-(6.51).

6.6 Conclusion

The optimization problem with quadratic criterion for linear multivariable control systems has been considered. The structure of the optimal control system with two controllers, whose performance is globally equivalent to the optimal regulator with single controller. The dimension of each optimal control subsystem is the same as the dimension of the overall optimal control system. However, each subsystem works in the corresponding orthogonal state subspace. Therefore, we can obtain the solution to the matrix Riccati equation for the subsystem much easier than the one for the overall system because the actual dimension needed to obtain the solution to the matrix Riccati equation for each subsystem is less than the dimension of the overall system.

Chapter 7

CONCLUSION AND FURTHER RESEARCH

In this dissertation, our attention is focused on the controller design for linear multivariable discrete-time systems and we have developed several controller design methods suitable for implementation by digital computers. The fundamental objective of controllers is to resolve a number of questions pertaining to the compensation of linear systems.

In many practical situations, the control actions are implemented to the systems subject to external uncertain factors — disturbances, parameter variations, monitoring error of the system output, partial failure of the system, and so on. The design methods of controllers developed in this dissertation are aimed to achieve prespecified performances and suppress undesirable effects due to uncertain factors.

This final chapter summarizes the results obtained in this dissertation and presents the topics for possible further research in this field.

7.1 Problems of the Design of Dead-Beat Controllers with Asymptotic Disturbance Rejection

The first part of this dissertation, consisting of Chapters 2 and 3, discussed design problems of dead-beat controllers with asymptotic disturbance rejection for linear multivariable discrete-time systems with disturbance input.

Chapter 2 considered the problem of output dead-beat control with asymptotic disturbance rejection by state feedback. By using an eigenvalue - generalized eigenvector assignment technique, a design algorithm of controller was proposed for computing the constant state feedback gain which forces the output zero in at most κ_1 (: the maximal value of the Kronecker invariants of the system) time steps and keeps it zero independent of disturbance input. The class of dead-beat controllers with asymptotic disturbance rejection was proved to be wider than the class of controllers derived from simultaneous disturbance localization and dead-beat control by pole assignment.

Chapter 3 discussed the problem of output dead-beat control with asymptotic disturbance rejection by dynamic compensation. In practical situations, the dynamic compensator is widely used to reconstruct the system state and determine the control input, because the complete system state is usually not directly available. We derived a characterization for the class of output dead-beat controllers independent of disturbance input.

The topics recommended for further research will be stated in the following.

1. Design of Output Dead-Beat Controllers with Asymptotic Disturbance Rejection by Output Feedback
2. Design of Minimum Time Output Dead-Beat Controllers with Asymptotic Disturbance Rejection
3. Design of Output Dead-Beat Controllers for Linear Discrete-Time Systems with Unstable Zeros by Output Feedback

The problems of pole assignment by output feedback [41, 31] and eigenvalue - eigenvector assignment using output feedback [84] has also attracted a considerable attention. But the problem of simultaneous pole placement and decoupling by output feedback was only partially solved [33, 74]. A characterization of all output dead-beat controllers with asymptotic disturbance rejection by output feedback is not obtained up to now.

The problem of minimum time control is one of the important topics in control theory [4]. The minimum time output dead-beat controller with asymptotic disturbance rejection is one of the ideal discrete-time controllers.

It is known that the output dead-beat control may give an unstable closed loop system if some of zeros of the system is located outside the unit circle. This means

that the closed loop discrete-time system is extremely sensitive to parameter variations [49]. When the system with disturbance input has zeros outside the unit circle, it is of interest to search for output feedback which forces the output of the system from any initial state x_0 to zero in finite time steps and thereafter keeps the output zero independent of disturbance input and which gives a stable closed loop system.

7.2 Problems of Adaptive Regulation for Linear Systems with Unknown Input

The second part of this dissertation, consisting of Chapters 4 and 5, discussed the design problem of output regulators using adaptive observers for linear discrete-time systems with unknown input, which is supposed to be generated as a state variable from a linear dynamic system with constant unknown system parameters.

Chapter 4 discussed the adaptive regulation problem for linear discrete-time systems with unknown input. We have proposed the design method of the dynamic controller which stabilize the closed loop of the system by using a constant feedback gain and compensate for the unknown input by using the unknown input observer through a linear functional observer and an adaptive state observer, and which forces the output of the system from any initial state x_0 to zero and thereafter keeps the output zero.

Chapter 5 considered the problem of efficient control with dual controllers for a single plant. The dual controllers consisting of the deterministic adaptive regulator (derived in Chapter 4) and the stochastic sub-optimal regulator are switched alternately in order to obtain the output of the plant with smaller errors.

In the control action using the dual controllers, when the plant begins to work, the stochastic sub-optimal regulator with time-varying feedback gain, which is calculated by off-line algorithm in advance, begins to control the plant first and the parameter estimation process begins to work at the same time. When the estimation error becomes sufficiently small, the controller is switched to the adaptive regulator with constant feedback gain and compensator begins to control the plant instead of sub-optimal regulator.

The topics recommended for further research will be stated in the following.

1. Adaptive Regulation for a Plant with Unknown Input Generated by a Dynamic System with Time-Varying Unknown Parameters
2. Optimal Estimation in Stochastic Optimal Regulator of Dual Controllers

The problem of adaptive regulation for plants with time-varying unknown parameters is one of the main themes of current research in adaptive control [73]. Although considerable attention has been given to this problem since the early days of adaptive control, it has only recently been studied on a rigorous basis [43, 53].

The problem of adaptive regulation for a plant with unknown input from a time-varying linear system is one of the interesting subjects in practical applications.

In the problem formulation of dual controllers, the control input derived from the stochastic sub-optimal regulator is composed of the optimal feedback gain and the sub-optimal estimated state, because the optimal estimated state can not be obtained by any conventional observers. It is an important subject to develop a scheme of optimal state estimation which can be used in the stochastic optimal regulator.

7.3 Problems of Optimal Regulation with Multiple Controllers

Chapter 6 discussed the optimization problem with quadratic criterion for linear multivariable control systems with multiple control agents. As the dimension of a dynamic system to be controlled becomes large, it is often too costly (sometimes practically impossible) to have only one decision maker (or controller) in the system who possesses all available information on the system and make all the decisions for the system, for example, the optimal control input. In order to avoid such a practical difficulty in implementation of control, we derived the new structure of the optimal control systems with multiple controllers, which is globally equivalent to the optimal control system with a single controller.

The topics recommended for further research will be stated in the following.

1. Optimal Control with the Performance Index in General Form

In the problem formulation of Chapter 6, we have only considered the limited case that the performance index to be minimized has a non-zero weighting matrix with respect to system state ($Q \neq 0$) and zero weighting matrix with respect to control input ($R = 0$). It is important to consider the optimal regulator by multiple controllers with the general form of performance index with non-zero weighting matrices with respect to state and control input ($Q \neq 0$, $R \neq 0$).

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